

Formulas, Inequalities, and Functions

$$z^{-1} = \frac{x - iy}{x^2 + y^2}$$

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$$

$$\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im} z = \frac{z - \bar{z}}{2i}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$z^n = z_0 = r_0 e^{i\theta_0} \iff z = \sqrt[n]{r_0} \exp \left[i \left(\frac{\theta_0}{n} + \frac{2\pi k}{n} \right) \right]$$

$$z^c = e^{c \log z}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \text{ and } \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$\sin^{-1} z = -i \log(iz + \sqrt{1 - z^2})$$

$$\arg z = \arg r e^{i\theta} = \theta$$

$$-\pi < \operatorname{Arg} z \leq \pi$$

$$\log z = \ln r + i(\theta + 2\pi k), \quad k \in \mathbb{Z}, \quad \operatorname{Log} z = \log z|_{k=0}$$

Complex Mappings

$f(z) = z + c$: **translation** by a constant

$f(z) = e^{i\alpha} z$: **rotation** by α

$f(z) = kz$: **magnification** leaving the argument invariant

Definition (Möbius transformation):

$$T : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}, \quad z \mapsto T(z) = \frac{az + b}{cz + d}$$

Theorem: The Möbius transformation maps circles to circles in $\bar{\mathbb{C}}$. It is a composition of a linear transformation, and inversion, and another linear transformation.

Definition (Cross-ratio):

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Definition: A function $f : D \rightarrow \mathbb{C}$ is called a conformal transformation if it is analytic with $f'(z) \neq 0$ for all $z \in D$.

Theorem: Let f be conformal and C_1, C_2 be two curves smooth at z_0 where they intersect with acute angle ψ . Then $f(C_1), f(C_2)$ intersect with the same angle.

Theorem: Let ϕ be harmonic on a simply connected R . Then ϕ remains harmonic under any conformal transformation.

Corollary: Let T be a conformal transformation and C be a smooth arc. Then if $\phi(x, y) = k$ on C , $\Phi(u, v) = k$ on $T(C)$.

Limits and Continuity

Definition: $f : \tilde{U}_\epsilon(z_0) \rightarrow \mathbb{C}$. Then the limit

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

exists if $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Theorem: Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then the limit,

$$\lim_{z \rightarrow z_0} f(z) = w_0 = u_0 + iv_0$$

exists if and only if,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$$

Theorem:

$$\lim_{z \rightarrow \infty} f(z) = w_0 \iff \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z \rightarrow z_0} f(z) = \infty \iff \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \rightarrow \infty} f(z) = \infty \iff \lim_{z \rightarrow 0} \frac{1}{f(\frac{1}{z})} = 0$$

Definition: A function f is called continuous at z_0 if $f(z_0)$ and $\lim_{z \rightarrow z_0} f(z)$ exist and

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Theorem: If f is nonzero and continuous at z_0 , then it is nonzero throughout a neighborhood containing z_0 .

Definition: A complex-valued function that is analytic except possibly for poles is meromorphic.

Theorem: Let f be analytic at z_0 with $f'(z_0) \neq 0$. Then an inverse function f^{-1} exists in a neighborhood of $f(z_0)$.

Parametric and Contour Integrals

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt$$

Definition: Curve with no self intersections $\rightarrow$ simple. Meet at endpoints $\rightarrow$ simple closed.

$$L(C) = \int_a^b |z'(t)| dt$$

Definition: Let $C : z(t), t \in [a, b]$ be a contour and $f(z)$ be piecewise continuous on C . Then the contour integral of f on C is,

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$$

Differentiation

Lemma: Differentiability implies continuity.

Theorem: If $f'(z)$ exists at $z_0 = (x_0, y_0)$, then,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and $f'(z_0) = u_x + iv_x$

Theorem: The first partial derivatives of $f(z)$ exist for all $U_\epsilon(z_0)$, are continuous, and satisfy the C.R.E, then $f'(z_0) = u_x + iv_x$ exists.

Lemma: The C.R.E. in polar coordinates are,

$$ru_r = v_\theta, \quad u_\theta = -rv_r \text{ and } f'(z(r, \theta)) = e^{-i\theta}(u_r + iv_r)$$

Definition: A function $f : D \rightarrow \mathbb{C}$ that is differentiable at each point in $U_\epsilon(z_0) \subset D$ is holomorphic in $U_\epsilon(z_0)$.

Definition: Holomorphic at every point of $\mathbb{C} \rightarrow$ entire.

Theorem: If $f : D \rightarrow \mathbb{C}$ and $\bar{f}$ are holomorphic everywhere in D , or $|f|$ is constant everywhere in D , then f is constant throughout D .

Theorem: If $f(z) = u(x, y) + iv(x, y)$ is holomorphic in D , then its real part u and imaginary part v are harmonic in D , that is, they satisfy $h_{xx} + h_{yy} = 0$.

Corollary: A holomorphic function on a simply connected domain has an antiderivative.

Theorem: Let R be simply connected and $u : R \rightarrow \mathbb{R}$ be harmonic. Then there exists an analytic $f : R \rightarrow \mathbb{C}$ such that $u = \Re f$.

Integral Tricks

Finitely many poles above the real axis: Draw a contour $C = C_R \cup [-R, R]$ and use Cauchy residue formula to evaluate.

Fourier Integrals: Possibly apply Jordan's Lemma

$$\int_{-\infty}^{\infty} f(x) \cos kx dx + i \int_{-\infty}^{\infty} f(x) \sin kx dx = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

Poles on the real axis: Modify the contour to circumvent the poles. Use Lemma about negative even powers.

Branches: Create a contour like the indented path and let the curve approach the branch. Use Lemma about $\lim_{z \rightarrow 0} zf(z)$.

Trigonometric integrals: On the unit circle,

$$\sin \theta = \frac{z - z^{-1}}{2i} \text{ and } \cos \theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{iz}$$

Boundary Value Problems

Lemma: Let R be simply connected and bounded by a simple closed C . Suppose ϕ is harmonic in R with $\phi(C) \leq M$ for some real M . Then $\phi(R) \leq M$, attaining its maximum on the boundary.

Theorem: Let R be simply connected and bounded by a simple closed C . If there exists a harmonic ϕ with prescribed $\phi(C)$, then ϕ is unique.

Integral Theorems and Lemmata

Theorem (ML): Suppose C is a contour of length L , and $f(z)$ on C is piecewise continuous and bounded by $M \geq 0$, then,

$$\left| \int_C f(z) dz \right| \leq ML$$

Lemma: For a complex-valued piecewise continuous $w(t)$, $t \in [a, b]$,

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt$$

Theorem: Support $f : D \rightarrow \mathbb{C}$ is continuous and the contour C is entirely in D , then the following are equivalent,

1. The antiderivative $F(z)$ exists in D and for any C from z_1 to z_2 , we have $\int_C f(z) dz = F(z_2) - F(z_1)$
2. For every closed C , we have $\int_C f(z) dz = 0$

Theorem (Cauchy): Let C be simple closed in D and $f : D \rightarrow \mathbb{C}$ be holomorphic with f' continuous on and within C , then, $\int_C f(z) dz = 0$.

Theorem (Cauchy-Goursat): Let C be simple closed in D and $f : D \rightarrow \mathbb{C}$ be holomorphic on and within C , then $\int_C f(z) dz = 0$.

Theorem: Let C be a closed contour in a simply-connected domain D , and $f : D \rightarrow \mathbb{C}$ be holomorphic in D , then, $\int_C f(z) dz = 0$.

Theorem: Let f be a holomorphic function on a multiply connected domain, and let C be a positively and C_i be n negatively oriented simple closed contours, with C_i inside of and disjoint from C , then,

$$\int_C f(z) dz + \sum_{i=1}^n \int_{C_i} f(z) dz = 0$$

Corollary: We can deform simple closed contours without changing the contour integral value, provided f is holomorphic across the deformation.

Theorem (Cauchy's integral formula): Let C be a positively oriented simple closed contour, and f be holomorphic everywhere inside and on C , then for any z_0 inside C ,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}$$

Theorem: If f is holomorphic at a point z_0 , then its derivatives of all orders exist and are holomorphic at z_0 .

Theorem (Morera): Let f be continuous on a domain D . If $\int_C f(z) dz = 0$ for every closed contour C , then f is holomorphic throughout D .

Theorem (Liouville): Every bounded entire function is constant.

Theorem (Cauchy's residue theorem): Let C be a positively oriented, simple closed contour. If f is an analytic function on and within C except at a finite number n of isolated singularities z_k , then,

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{res } f(z_k)$$

Integral Theorems and Lemmata

Theorem: Suppose f is analytic throughout $\mathbb{C}$ except for isolated singularities which are all enclosed by the positively oriented simple closed C ,

$$\int_C f(z) dz = -2\pi i \text{res } f(\infty) = 2\pi i \text{res } \left(\frac{1}{z^2} f\left(\frac{1}{z}\right) \right) (0)$$

Lemma (Jordan): Suppose $f(z)$ is analytic throughout the upper half plane and outside the origin-centered circle of radius R_0 , and consider the half circle contour $C_R : |z| = R > R_0$, $0 \leq \theta \leq \pi$. If, $|f(z)|(C_R) \leq M_R$ and $\lim_{R \rightarrow \infty} M_R = 0$ then for every $k > 0$,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{ikz} dz = 0$$

Lemma: Suppose f is analytic in $0 < |z - x_i| < r$ and let C_i be the clockwise semicircle $z = x_i + r_i e^{i\theta}$, $\pi \geq \theta \geq 0$ with $r_i < r$. Then if the Laurent series of f about x_i contains no even negative powers, then we have,

$$\lim_{r_i \rightarrow 0} \int_{C_i} f(z) dz = -i\pi \text{res } f(x_i)$$

Lemma: Let C_r be a circular segment of radius r centered at $z_0 = 0$ and $f(z)$ be continuous on C_r , then,

$$\lim_{z \rightarrow 0} z f(z) = 0 \text{ implies } \lim_{r \rightarrow 0} \int_{C_r} f(z) dz = 0$$

Formula: Laplace transform of a continuous real-valued function $f : t \in \mathbb{R}_0^+ \mapsto f(t)$

$$\bar{f} : s \in \mathbb{C} \mapsto \int_0^\infty f(t) e^{-st} dt$$

Formula (Bromwich Formula):

$$f(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R} \bar{f}(s) e^{st} ds$$

Singularities and Residues

- **Isolated singularity:** f is analytic in a deleted neighborhood $\tilde{U}(z_0)$
- **Essential isolated singularity:** There is no positive integer m such that $c_n = 0$ for all $n < -m$.
- **Pole of order m :** Exists a positive integer m s.t. $c_n = 0$ for all $n < -m$.
- **Removable singularity:** $c_n = 0$ for all $n < 0$. Can be removed by defining $f(z_0) = c_0$

Definition: The residue of f at z_0 is the coefficient of $\frac{1}{z-z_0}$, the number

$$\text{res } f(z_0) = c_{-1} = \frac{1}{2\pi i} \int_C f(z) dz$$

Lemma:

$$\text{res } f(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)](z_0)$$

Theorem: A zero of an analytic f that is not identically zero is isolated.

Lemma: Suppose $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, analytic, then,

$$\text{res } \left[\frac{p}{q} \right] (z_0) = \frac{p(z_0)}{q'(z_0)}$$

Sequences and Series

Definition: An infinite sequence $\mathbb{N} \rightarrow \mathbb{C}$, $\{z_n\} = \{z_1, z_2, z_3, \dots\}$ has a limit z is for every $\epsilon > 0$ there exists a positive integer N_ϵ s.t.

$$n > N_\epsilon \implies |z_n - z| < \epsilon$$

Definition: A function $f : U_\epsilon(z_0) \rightarrow \mathbb{C}$ is called analytic if for every $z \in U_\epsilon(z_0)$, it is represented by a convergent power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem: If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for $z = z_1 \neq z_0$, then it is absolutely convergent everywhere in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$. The greatest circle centered at z_0 such that the power series converges everywhere inside is called the circle of convergence.

Theorem: Suppose z_1 is inside the circle of convergence $|z - z_0| < R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Then the series is uniformly convergent in the closed disk $|z - z_0| \leq R_1$ where $R_1 = |z_1 - z_0|$.

Theorem: A power series is a continuous function $S(z)$ everywhere inside its circle of convergence $|z - z_0| = R$.

Theorem (Taylor): Suppose f is holomorphic throughout the open disk $|z - z_0| < R_0$. Then f is analytic, given by the convergent infinite power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Theorem (Laurent): Suppose f is analytic in an annular domain $R_0 < |z - z_0| < R_1$, and let C be a simple closed contour in this domain. Then throughout the domain, f has the convergent series,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}$$

Theorem: A power series that converges to $f(z)$ everywhere inside $|z - z_0| = R$ is the Taylor series of f about z_0 .

Theorem: A series

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

convergent to $f(z)$ everywhere in the annular domain around z_0 is the Laurent series of f about z_0 .

Miscellaneous Theorems

Theorem (Argument principle): For a meromorphic function f ,

$$Z - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi} \Delta_C \arg f = \nu(\Gamma, 0)$$

Theorem (Rouché): Suppose f and g are analytic on and inside a simple closed contour C , and $|g(z)| < |f(z)|$ on C . Then f and $f + g$ have the same number of zeros inside C .

Theorem (Brouwer): Let $D : |z| \leq 1$ and $g : D \rightarrow \text{int } D$ be analytic with $|g(z)| < 1$. Then g has a fixed point z_0 such that $g(z_0) = z_0$.

Theorem (Inverse function): Let f be analytic at z_0 with $f'(z_0) \neq 0$. Then its inverse f^{-1} in a neighborhood $D \ni w_0 = f(z_0)$ is unique and analytic.