CHROMATIC SYMMETRIC FUNCTIONS

by

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ABSTRACT

Stanley [14] introduced the *Chromatic Symmetric Function* (CSF) in 1995 as a symmetric function associated to a graph $G = (V, E)$ and defined via the proper colorings of G. One major open question surrounding the CSF is whether it can distinguish tree graphs (up to isomorphism), and it has been verified to be true for trees up to 29 vertices [9]. In **Chapter 1**, we give explicit formulas for the CSF of complete bipartitie graphs, windmill graphs, and lollipop graphs. **Chapter 2** serves as a review of the graph and tree properties known to be preserved by the CSF. Lastly, the major results of this paper are in **Chapter 3**, where we give simple, combinatorial proofs that certain infinite families of trees, namely path graphs with one leaf adjoined, and spiders, are distinguished by their CSFs. Because our proofs are combinatorial in-nature and not algebraic, they provide a unique insight into the structure of the CSF. Furthermore, we give a classification of the difference of two CSFs of trees in-terms of forest graphs, which implies a new approach to the proof that trees are distinguished by the CSF.

Chapter 1

A BACKGROUND ON (CHROMATIC) SYMMETRIC FUNCTIONS

1.1 Introduction and Results

The study of symmetric functions is not new, it dates back at least to the mid- to late-nineteenth century [5], but it remains a very active area of research in algebraic combinatorics today. Abstracting a system where symmetry is present to a representative symmetric function is useful to apply the many powerful transformations in symmetric function theory, and then interpret those new results in the context of the original system. The chromatic symmetric function (CSF) is an example of this, exploiting the inherent symmetry in proper graph colorings. Despite its introduction 30 years prior, the CSF remains a highly active area of research today, fueled by its open problems which are easily stated, but are very difficult to solve.

In **Chapter 1**, we give an overview of symmetric functions, including definitions of various common bases. Furthermore, in this chapter we introduce the definition of the chromatic symmetric function, as well as the most important combinatorial properties of the CSF which will be used throughout the entire paper. We also introduce the special topic of chromatic bases. Special attention should be given to **Theorem 1.3.1.** and **Theorem 1.3.2**,[1] which are original to this paper and give explicit expressions for the CSF of particular families of graphs, namely complete bipartitie graphs, windmill graphs, and lollipop graphs.

Chapter 2 serves as a review of the most notable graph properties which are known to be preserved by the CSF. In particular, we give original proofs for several of these properties. We give a particular focus to properties which are preserved in the case where the CSF is known to correspond to a tree, where we give high-level overviews of the relevant proofs.

Chapter 3 covers an open problem known as *Stanley's isomorphism conjecture*, namely, that the CSF distinguishes tree graphs. We begin by examining the simple family of graphs obtained by adjoining a single leaf to a path graph. **Theorem 3.1.1** shows that the intuitive approach of studying the algebraic difference in two CSFs is sufficient to recover the difference in the structure of the graphs themselves. Next, **Theorem 3.2.1** gives a simple, combinatorial argument that trees with at most one vertex of degree greater than 3 can be reconstructed only using their CSF. Finally, we give a new approach to the proof of Stanley's conjecture in **Theorem 3.3.2**, which is related to the result of **Theorem 3.3.1**, which states the algebraic difference of two tree CSFs is an algebraic combination of CSFs of forests. In particular, we outline an inductive proof of Stanley's conjecture, and leave it as future work to complete this argument.

[1] In general, we reserve the word **Theorem** for results which are original to this work.

This paper is written at the undergraduate level, and assumes only a basic knowledge of graph theory and standard notations, as well as combinatorial reasoning.

1.2 Symmetric Functions and Integer Partitions

A *symmetric function* is, intuitively, a polynomial expression in an infinite amount of commutative indeterminates, with the special property of invariance under any permutation of its variables. To be more formal, let $\pi : \mathbb{N} \to \mathbb{N}$ be any permutation (i.e., π is a bijective mapping) and $f(x_1, x_2, ...) \in \mathbb{Q}[x_1, x_2, ...]$. [2] Then we say f is a symmetric function if,

$$f(x_1, x_2, ...) = f(x_{\pi(1)}, x_{\pi(2)}, ...)$$

For example, the function,[3]

$$g(x_1, x_2, ...) = \prod_{(i,j)\in\mathbb{N}\times\mathbb{N}} x_i^2 x_j$$

is symmetric, and includes monomial terms such as $x_{42}^2 x_{15}$ and x_{27}^3. For convenience, will usually omit the argument and denote the symmetric function $f(x_1, x_2, ...)$ simply as f. In fact, as long as it is "large enough", the number of indeterminates considered in a symmetric function is usually irrelevant.

1.2.1 Integer Partitions

Integer partitions are essential to understand symmetric functions, as symmetric function bases homogeneous of degree n are defined over the set of all partitions of n. First, we define integer partitions, and then several operations on integer partitions which will be useful later-on in this chapter.

Let $n \in \mathbb{Z}^+$. We say that $\lambda \vdash n$ is an *integer partition* of n and write $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ if,

1. $\lambda_i \in \mathbb{Z}^+$ for all $1 \leq i \leq k$
2. $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ (weakly decreasing)
3. $\lambda_1 + \lambda_2 + ... + \lambda_k = n$

In this case, we let define $l(\lambda) = k$ to be the *length* of λ. For convenience, if $\lambda_i = \lambda_{i+1} = ... = \lambda_j$ for $i < j$, then we write $\lambda = (\lambda_1, ...\lambda_{i-1}, \lambda_i^{j-i+1}, \lambda_{j+1}, ...\lambda_{l(\lambda)})$. For a fixed n, we let $p(n)$ to be the number of integer partitions of n, called the *partition number* of n. Interestingly, there is no known closed-form expression for $p(n)$. Next, we define several operations on integer partitions, but note that the notation may not be standard.

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$ be a partition of n. We say that $\mu \subset \lambda$ (μ is a subpartition of λ) if $\mu = (\lambda_{\kappa(1)}, \lambda_{\kappa(2)}, ..., \lambda_{\kappa(\ell)})$, where $\ell \leq k$, for some strictly increasing function $\kappa : \{1, 2, ..., \ell\} \to \{1, 2, ..., l(\lambda)\}$. Moreover, we let $\lambda - \mu$ to be the partition λ with the elements of μ removed, that is $\lambda - \mu$ contains λ_i as a component if and only if $i \neq \kappa(s)$ for any $s \in \{1, 2, ..., \ell\}$. Similarly, we let $\lambda + \mu$ to be partition obtained by sorting the multiset $\{\mu_i : 1 \leq i \leq l(\mu)\} \cup \{\lambda_j : 1 \leq j \leq l(\lambda)\}$ in weakly decreasing order. If we let $m = \sum \mu_i$, then it is not hard to see that $\mu \vdash m$, $\lambda - \mu$ is

[2] For our purposes, we take the coefficients of each monomial in f to be from the field $\mathbb{Q}$, unless otherwise specified. That said, symmetric functions are well-defined for any choice of a commutative ring of coefficients. $\mathbb{Z}$ is often considered as a ring of coefficients over which to define the ring of symmetric functions.

[3] One must be careful to distinguish between a "function" in the sense of a mapping from inputs to outputs, and a "function" in the symmetric function sense, which is a representation of a formal power series.

a partition of $n - m$, and $\lambda + \mu$ is a partition of $n + m$. Lastly, if $\lambda = (n^{r_n}, (n-1)^{r_{n-1}}, ..., 2^{r_n}, 1^{r_1})$ where each $r_i \geq 0$ for $1 \leq i \leq n$, we define $\widetilde{\lambda} = r_1! r_2! ... r_n!$. Notice that in this case, the r_i's are not exponents in the traditional (real number) sense, but count the number of times i appears in the partition λ.

Finally, we define the lexicographic ordering on the set of partitions of a fixed integer n, which is a total ordering. Namely, for $\lambda, \mu \vdash n$ such that $\lambda \neq \mu$, let i be the smallest index for which $\lambda_i \neq \mu_i$ (if there is no such index, then $\lambda = \mu$). Then, we say $\lambda <_L \mu$ if $\lambda_i < \mu_i$, and otherwise $\mu <_L \lambda$.

Example 1.2.1. The partitions of $n = 5$ are,

$$\begin{array}{cccc} \lambda^1 = (5) & \lambda^2 = (4,1) & \lambda^3 = (3,2) & \lambda^4 = (3,1,1) \\ \lambda^5 = (2,2,1) & \lambda^6 = (2,1,1,1) & \lambda^7 = (1,1,1,1,1) & \end{array}$$

Hence, $p(5) = 7$. We can see that $\lambda^i >_L \lambda^{i+1}$ for all $1 \leq i < 7$. Moreover, we have that $\mu = (2,1) \subset \lambda^6$, $\lambda^6 - \mu = (1,1)$, and $\lambda^6 + \mu = (2^2, 1^4)$. Lastly, we have that $\widetilde{(\lambda^6 + \mu)} = 2! \cdot 4! = 48$. □

1.2.2 Symmetric Function Bases

Let Λ^n denote the set of symmetric functions homogeneous of degree n, indicating that for any $f \in \Lambda^n$, the sum of the exponents in each monomial in f is n. It is well-known that Λ^n is a finite-dimensional vector space over $\mathbb{Q}$ with dimension $p(n)$ (or, more generally, Λ^n is a $\mathbb{Q}$-module), where $p(n)$ is the *partition number* of n. There are five commonly-used bases of Λ^n. We give the definition and some properties of each of these bases here.

Monomial Basis Let $\lambda \vdash n$ be a partition of n. The monomial symmetric function m_λ is defined as,

$$m_\lambda = \sum_{\alpha=(\alpha_1,...,\alpha_k)\sim\lambda} \sum_{j_1<...<j_k} x_{j_1}^{\alpha_1} \cdot ... \cdot x_{j_k}^{\alpha_k}$$

where $\alpha \sim \lambda$ if α is a rearrangement of the parts of λ. That is, $\alpha = (\lambda_{\pi(i)})_{i=1}^{l(\lambda)}$ where $\pi : \{1, 2, ..., l(\lambda)\} \to \{1, 2, ..., l(\lambda)\}$ is a permutation. Such an α is commonly referred to as a *composition* of n, i.e. a partition irrespective of any ordering of its parts.

We will show that the set m_λ over all partitions $\lambda \vdash n$ is indeed a basis for Λ^n, proving that the dimension of Λ^n is $p(n)$.

Lemma 1.2.1. *The set* $\mathbf{m}_n = \{m_\lambda : \lambda \vdash n\}$ *of monomial symmetric functions homogeneous of degree* n *is a basis for* Λ^n.

Proof. First, we will show linear independence of $\mathbf{m}_n$. For any monomial of the form $x_1^{\lambda_1} x_2^{\lambda_2} ... x_k^{\lambda_k}$, such that $\sum_{i=1}^k \lambda_i = n$, there is only one partition of n whose parts are $(\lambda_1, \lambda_2, ..., \lambda_k)$, up to rearrangement, call it λ, and the monomial is a term in m_λ. Hence, for any two partitions, $\lambda \neq \alpha$, m_λ and m_α have no terms in-common. In other words, $\sum_{\lambda\vdash n} a_\lambda m_\lambda = 0$ if and only if $a_\lambda = 0$ for every $\lambda \vdash n$. Hence, the set $\mathbf{m}$ is linearly independent.

Next, we will show that the set $\mathbf{m}_n$ spans Λ^n. Let $f \in \Lambda^n$. We will argue by induction on the number of terms in f. If $f = 0$, then $f = 0 \cdot m_\lambda$ for any $\lambda \vdash n$. Suppose $f \neq 0$, then f contains a term $a_\mu \cdot x_1^{\mu_1} x_2^{\mu_2} ... x_\ell^{\mu_\ell} \in a_\mu m_\mu$ where $\mu = (\mu_1, \mu_2, ..., \mu_k)$. Furthermore, $f - a_\mu m_\mu \in \Lambda^n$ and has fewer terms than f. By induction on the number of terms in f, it follows that we can express f as a linear combination of monomial symmetric functions in $\mathbf{m}_n$. □

On the surface, this result seems trivial, but turns-out to be very useful to show that the other symmetric function bases in this chapter are indeed bases of Λ^n over $\mathbb{Q}$. That is, we can write each of the basis elements as a linear combination of monomial symmetric functions (see Chapter 7 in [15] or Chapter 1.5 in [11] for explicit transition formulas), and cite **Lemma 1.2.1** to complete the argument.

Elementary and Homogeneous Bases The *elementary* symmetric function basis is the collection of terms, $\mathbf{e}_n = \{e_\lambda = e_{\lambda_1} \cdot \ldots \cdot e_{\lambda_{l(\lambda)}} \mid \lambda \vdash n\}$, where,

$$e_k = m_{(1^k)} = \sum_{j_1<j_2<\ldots<j_k} \prod_{i=1}^{k} x_{j_i}$$

That $\mathbf{e}_n$ indeed forms a basis for Λ^n is referred to as the **Fundamental Theorem of Symmetric Functions.** This is due to the fact that, along with the basis property of $\mathbf{e}_n$ the set $\{e_n\}_{n\geq 1}$ is algebraically independent, that is, there is no nonzero polynomial $g(\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $g(e_1, e_2, \ldots, e_n) = 0$. The result is the set of all symmetric functions is the set of all polynomials with coefficients in $\mathbb{Q}$ over $\{e_n\}_{n\geq 1}$, or, $\Lambda = \mathbb{Q}[e_1, e_2, \ldots]$.

Next, the *complete homogeneous* symmetric function basis is the collection of terms $\{h_\lambda = h_{\lambda_1} \cdot \ldots \cdot h_{\lambda_{l(\lambda)}} \mid \lambda \vdash n\}$, where,

$$h_k = \sum_{j_1\leq j_2\leq\ldots\leq j_k} \prod_{i=1}^{k} x_{j_i}$$

Example 1.2.2. Terms such as $x_1x_2x_3x_4$ and $x_3x_{20}x_{32}x_{46}$ are monomials in both e_4 and h_4. However, the terms $x_1^2x_4x_5$ and x_6^4 are in h_4 but not in e_4. □

We are able to derive interesting, nontrivial properties about the various symmetric function bases by considering their ordinary generating functions. For example, the ordinary generating function for the sequence of elementary symmetric functions $\{e_n\}_{n\geq 0}$ is given by,

$$E(t) = \sum_{n\geq 0} e_n t^n = (1 + x_1 t) \cdot \ldots \cdot (1 + x_k t) \cdot \ldots = \prod_{i\in\mathbb{N}} (1 + x_i t)$$

Likewise, it is not hard to see that the ordinary generating function for the homogeneous symmetric functions $\{h_n\}_{n\geq 0}$ is,

$$H(t) = \sum_{n\geq 0} h_n t^n = (1 + x_1 t + x_1^2 t^2 + \ldots) \cdot \ldots \cdot (1 + x_k t + x_k^2 t^2 + \ldots) \cdot \ldots = \prod_{i\in\mathbb{N}} \frac{1}{1 - x_i t}$$

And hence, remarkably, we discover the relation $H(t)E(-t) = 1$, therefore,

$$\sum_{j=0}^{n} (-1)^j e_j h_{n-j} = 0$$

Furthermore, we define the *usual involution* ω on symmetric functions by $\omega(e_\lambda) = h_\lambda$ for any $\lambda \vdash n$. It is interesting to examine the properties of the involution ω on symmetric functions whose coefficients have combinatorial interpretations. For example. in the case of the chromatic symmetric function (CSF), applying the involution ω transforms the CSF to another symmetric function whose coefficients give information about the *acyclic orientations* of the original graph.

Lastly, as an example of the statement from the end of the previous section about expressing the basis elements of different symmetric function bases in-terms of the monomial symmetric functions, we give the following lemma, which also reappears later in this chapter.

Lemma 1.2.2 ([15]). *Let $\lambda = (\lambda_1, \dots, \lambda_k), \mu = (\mu_1, \dots, \mu_\ell) \vdash n$ be partitions and $M_{\lambda,\mu}$ the number of $0,1$-matrices with row sums λ_i and column sums μ_j, then,*

$$e_\mu = \sum_{\lambda \vdash n} M_{\lambda,\mu} m_\lambda$$

Power Sum Basis The *power sum* symmetric function basis is defined by $\{p_\lambda = p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_{l(\lambda)}} \mid \lambda \vdash n\}$, where,

$$p_k = \sum_{i \in \mathbb{N}} x_i^k$$

Furthermore, the ordinary generating function for the sequence $\{\frac{p_n}{n}\}_{n \geq 0}$

$$P(t) = \sum_{n \in \mathbb{N}} \frac{p_n}{n} t^n = \log\left(\prod_{j \in \mathbb{N}} \frac{1}{1 - x_j t}\right) \tag{1.1}$$

Proof. By Taylor expansion, $\log\left(\frac{1}{1-x}\right) = \sum_{n \in \mathbb{N}} \frac{1}{n} x^n$. Hence,

$$P(t) = \sum_{n \in \mathbb{N}} \frac{p_n}{n} t^n = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{N}} \frac{1}{n} (x_j t)^n = \sum_{j \in \mathbb{N}} \log\left(\frac{1}{1 - x_j t}\right) = \log\left(\prod_{j \in \mathbb{N}} \frac{1}{1 - x_j t}\right)$$

□

More generally, we have the relation,

$$\log\left(\prod_{i,j} (1 - x_i y_j)^{-1}\right) = \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y)$$

where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. In particular, we notice the expression for the generating function $H(t)$ of the homogeneous symmetric functions appear in the expression (1.1). Based on this observation and with a little more manipulation, we derive,

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda, \text{ and by applying } \omega, \ e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$$

where $\varepsilon_\lambda = (-1)^{n - l(\lambda)}$ and z_λ is the number of permutations of $\sigma \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters, such that σ commutes with a fixed $w_\lambda \in \mathfrak{S}_n$ with cycle type λ, namely, the blocks of w_λ in reduced form have lengths corresponding to the components of λ. We direct the reader to Chapter 7 of [15] for a full derivation. Indeed, generating functions are one of the primary tools to discover relationships between symmetric function bases, among a host of other interesting applications.

Schur and Skew Schur Symmetric Functions The Schur and Skew Schur symmetric functions are defined via fillings of *Young diagrams* (also known as Young tableaux, or Ferrers diagrams), which have a deep connection to the mathematical study of representation theory, particularly the representations of the symmetric group. We begin by giving an overview of Young tableaux. Then, we define the Schur and Skew Schur functions.

A *Young diagram* is a representation of a partition $\lambda \vdash n$ as a left-justified collection of boxes, with the number of boxes in each row corresponding to one component of λ, and the rows are weakly decreasing in size from top to bottom. An example of a Young diagram for the partition $\lambda = (4,2,2,1)$ is given in **Figure 1.1**.

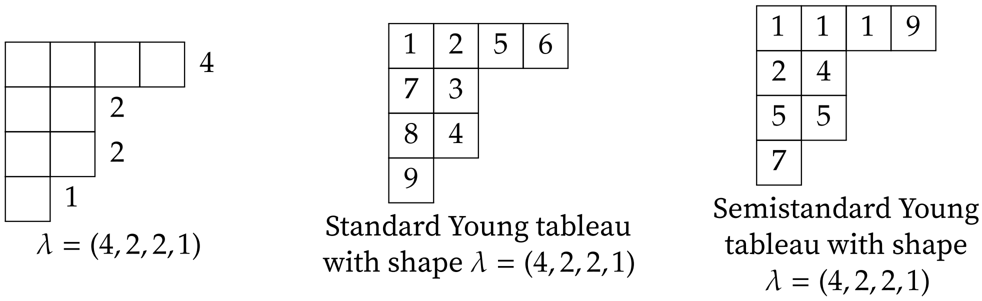


Figure 1.1: Examples of standard and semistandard Young tableaux.

The Schur symmetric functions can be defined combinatorially in-terms of Young tableaux, although there are many other ways to define the Schur functions, such as in a purely algebraic way. A standard Young tableau of a Young diagram with shape λ is a way of assigning a unique number from $\{1,2,\ldots,n\}$ to each box such that the entries are strictly increasing from left to right and from top to bottom. An example of a standard Young tableau with shape $\lambda = (4,2,2,1)$ is given in **Figure 1.1**. In total, there are 180 standard Young tableaux with shape λ

A related concept is that of *semistandard* Young tableaux, which is defined in the same way as standard Young tableaux, except the entries are *strictly* increasing from left to right, while they can be weakly increasing from top to bottom. The infinite set of semistandard Young tableaux of shape λ with entries in $\mathbb{N}$ (with repeats allowed) is denoted as $\mathrm{SSYT}(\lambda)$. Alternatively, we can restrict the entries for the fillings to come from another partition $\mu \vdash n$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, indicating that there are μ_i boxes filled with the number i. In this case, we let $\mathrm{SSYT}(\lambda, \mu)$ refer to the (finite) set of semistandard Young tableaux with shape λ and fillings from μ. The Schur symmetric functions are defined as,

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T$$

where x^T denotes the monomial $\prod_{i \in \mathbb{N}} x_i^{T(i)}$ where $T(i)$ is the number of i's that appear in the semistandard tableau T. For example, if T is the semistandard tableau in **Figure 1.1**, then $x^T = x_1^3 x_2 x_4 x_5^2 x_7 x_9$. Moreover, the set $\{s_\lambda : \lambda \vdash n\}$ forms a $\mathbb{Q}$-basis for Λ^n.

The *skew Schur* functions are defined similarly, except they are indexed by the skew partitions λ/μ where $\mu \leq \lambda$ (μ is contained in λ) indicating that if $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $m \leq n$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq m$. The shape of λ/μ is the shape of λ overlayed with the shape of μ, deleting all the boxes that intersect. And example of λ/μ is given in **Figure 1.2** where $\lambda = (5,4,4,2,1)$ and $\mu = (4,3,2)$.

We now define,

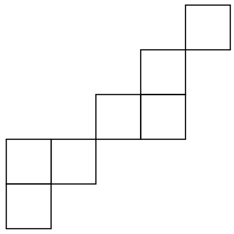


Figure 1.2: Shape of λ/μ where $\lambda = (5, 4, 4, 2, 1)$ and $\mu = (4, 3, 2)$.

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T$$

Young tableaux are one of the main objects of study in algebraic combinatorics, and have many nice combinatorial properties. Previously, we had mentioned that there were 180 standard Young tableaux with shape $(4, 2, 2, 1)$. One question that arises is whether there is a simple way to count the number of standard Young tableaux of a certain type. In fact, there is a beautiful formula, discovered by Frame, Robinson, and Thrall [16], called the *hook length formula*, which does exactly that. We end our discussion of symmetric function bases by explaining the hook length formula.

Let $\lambda \vdash n$ be a partition and let $[\lambda]$ denote the set of squares in the Young diagram with shape λ. We label each box of $[\lambda]$ with an ordered pair (i, j) in the usual sense, with $(0, 0)$ corresponding to the top, leftmost box, with i increasing as we move to the right and j increasing as we move down. An example of the ordering of the partition $\lambda = (5, 4, 4, 2, 1)$ is given in **Figure 1.3**.

(0,0)	(1,0)	(2,0)	(3,0)	(4,0)
(0,1)	(1,1)	(2,1)	(3,1)	
(0,2)	(1,2)	(2,2)	(3,2)	
(0,3)	(1,3)			
(0,4)				

Figure 1.3: Ordering of $[\lambda]$ where $\lambda = (5, 4, 4, 2, 1)$.

We then define then *hook* of a box (i, j) to be the boxes extending strictly to the left of (i, j) and strictly below (i, j) that is, the set $h_{i,j} = \{(x, y) \mid x \geq i \ \&\ y \geq j\}$ and the *hook length* to be $\#h_{i,j}$. An example of a hook and hook length are given in **Figure 1.4**. Let f^λ denote the number of standard Young tableaux of shape λ, then the *hook length formula* states,

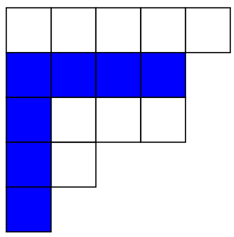


Figure 1.4: Hook of $(0, 1)$ in $[\lambda]$ where $\lambda = (5, 4, 4, 2, 1)$. The hook length is $\#h_{0,1} = 7$.

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in [\lambda]} \#h_{i,j}} \tag{1.2}$$

There are many proofs of the hook length formula, but one that is particularly elegant from Greene, Nijenhuis, and Wilf [7] designs a randomized algorithm as follows: We first pick a box (i, j) in the Young diagram at random, and then pick a box in $h_{i,j}$. We continue this process until the selected box lies on the right-most or bottom-most boundary of the diagram. In particular, let μ be the partition corresponding to the Young diagram with the box (i_0, j_0) residing on the right-most or bottom-most boundary removed. Then, the probability that the algorithm terminates on (i_0, j_0) is exactly f^μ/f^λ. The result follows by noticing that $f^\lambda = \sum_{\mu\uparrow\lambda} f^\mu$ and by normalization of probability measures, where $\mu \uparrow \lambda$ denotes that μ is constructed from λ by removing a boundary square, as previously stated.

1.3 Chromatic Symmetric Functions

The chromatic symmetric function (CSF) was introduced by Richard Stanley in his 1995 seminal work, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph* [14]. Given a graph $G = (V, E)$, the CSF of G is denoted as X_G and is defined as a sum over all proper colorings $\phi : V \to \mathbb{N}$ of G. For readers who are unfamiliar, a *proper coloring* ϕ of G is a labeling, or "coloring", of the vertex set of G such that if $(u, v) \in E$, then $\phi(u) \neq \phi(v)$. We now define,

$$X_G = \sum_\phi \mathbf{x}^\phi = \sum_\phi \prod_{v\in V} x_{\phi(v)}$$

It is not hard to justify that X_G is indeed a symmetric function. Namely, we can identify the vertices with the same labeling in any proper coloring, and permute the colorings associated to each of these sets to obtain another proper coloring. Indeed, permuting the colors associated to each set has the effect of permuting the subscript of the terms in each monomial.

Next, we review some basic properties of the chromatic symmetric function that will be fundamental to our later work. Firstly, we have the following result.

Lemma 1.3.1. *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be graphs, and let $G + H$ denote the disjoint union of G and H, then,*

$$X_{G+H} = X_G \cdot X_H$$

Proof. Let $\phi_G : V_1 \to \mathbb{N}$ and $\phi_H : V_2 \to \mathbb{N}$. Let $\phi_{G+H} : V_1 \cup V_2 \to \mathbb{N}$ be the unique mapping for which $\phi_{G+H}|_{V_1} \equiv \phi_G$ (the restriction of ϕ_{G+H} to the domain V_1) and $\phi_{G+H}|_{V_2} \equiv \phi_H$. Then ϕ is a proper coloring of $G + H$ if and only if ϕ_G and ϕ_H are proper colorings of G and H, respectively. Hence,

$$X_G \cdot X_H = \sum_{\phi_G,\phi_H} \mathbf{x}^{\phi_G}\mathbf{x}^{\phi_H} = \sum_{\phi_{G+H}} \mathbf{x}^{\phi_{G+H}} = X_{G+H}$$

□

In the next section, we wish to express the chromatic symmetric function in different symmetric function bases by finding a combinatorial interpretation of its coefficients.

1.3.1 Combinatorial Interpretations of the CSF

Monomial Basis The easiest and most intuitive basis by which to expand the CSF is the monomial basis. We first define the *augmented monomial symmetric functions* as $\widetilde{m}_\lambda = \tilde{\lambda} m_\lambda$. We are now ready to present the interpretation of the CSF of a graph in the (augmented) monomial basis.

Lemma 1.3.2. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n = \#V$ and a_λ be the number of partitions of V into components of size $\lambda_1, \lambda_2, ..., \lambda_k$ such that there are no edges between any two vertices in the same component.*[4] *Then,*

$$X_G = \sum_{\lambda \vdash n} a_\lambda \widetilde{m}_\lambda$$

Proof. The coefficient of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} ... x_k^{\lambda_k}$ is the number of ways to choose a partition of V into components of size $\lambda_1, \lambda_2, ..., \lambda_k$ such that there are no edges between any two vertices in the same component, and then color a component of size λ_i with the color i for each $i \in \{1, ..., k\}$. Let r_i be the number of components with size λ_i, then there are $r_i!$ ways to color the vertices in the components with size λ_i. □

Therefore, we can reduce the problem of computing the CSF of a graph to counting the number of partitions of V into independent sets type $\lambda \vdash n$. As it turns out, for certain families of graphs, such as complete bipartite graphs and windmill graphs, this counting is relatively simple, and allows us to compute the CSF easily (see **Theorem 1.3.1**).

Power Sum Basis Like the monomial basis, expressing the CSF in the power sum basis also has a simple combinatorial interpretation, which is particularly useful in proving theorems about the CSF. In this section, we outline Stanley's original proof of this combinatorial interpretation, as well as a more modern proof based on the deletion-contraction rule for a generalization of the CSF to weighted graphs. As a bit of notation, if $G = (V, E)$ is a graph and $S \subset E$, then we denote by $G(S) = (V, S)$ the subgraph of G with edge set S. Moreover, we denote by $\lambda_G(S)$ the partition of $n = |V|$ formed by sorting the number of vertices in each connected component of $G(S)$ in weakly decreasing order. Where G is obvious, we omit from the subscript and simply write $\lambda(S)$. We are now ready to state the result.

Lemma 1.3.3 ([14]).

$$X_G = \sum_{S \subset E} (-1)^{|S|} p_{\lambda(S)}$$

Proof. Fix $S \subset E$. We have that,

$$p_{\lambda(S)}(\mathbf{x}) = \sum_{\phi \in K_S} \mathbf{x}^\phi$$

where K_S is the set of all colorings which color the vertices within the same connected components of $G(S)$ with the same color. Hence,

$$\begin{aligned} \sum_{S \subset E} (-1)^{\#S} p_{\lambda(S)}(\mathbf{x}) &= \sum_{S \subset E} (-1)^{\#S} \sum_{\phi \in K_S} \mathbf{x}^\phi \\ &= \sum_{\phi} \mathbf{x}^\phi \sum_{S \subset E_\phi} (-1)^{\#S} \end{aligned}$$

[4] Such a set of independent vertices is, intuitively, called an *independent set*. If the sizes of a partition V into independent sets correspond to the parts of λ, we call it a partition of V into independent sets of *type* λ.

where the first sum is over all colorings (proper or not) $\phi : V \to \mathbb{N}$ and the set E_ϕ refers to the set of edges with both endpoints colored the same in ϕ. Suppose that for a coloring ϕ_0, E_{ϕ_0} is nonempty, then,

$$\sum_{S \subset E_\phi} (-1)^{\#S} = \sum_{k=0}^{\#E_{\phi_0}} (-1)^k \binom{\#E_{\phi_0}}{k} = 0$$

by symmetry of binomial coefficients. Hence, the double sum (1.3) selects only the colorings ϕ that are proper, which is exactly the definition of X_G. □

Before we give an alternative proof of this result, we first introduce the *vertex weighted CSF* as defined by Logan and Spirkl (2020) [4].

Definition 1.3.1. *Given a weighted graph $G = (V, E, \omega)$ with $\omega : V \to \mathbb{N}$, we define the vertex weighted chromatic symmetric function of G as,*

$$X_{(G,\omega)} = \sum_{\phi} \prod_{v \in V} x_{\phi(v)}^{\omega(v)}$$

where the sum is over all proper colorings $\phi : V \to \mathbb{N}$.

Furthermore, the main contribution of Crew and Spirkl was to provide a deletion-contraction rule for the vertex weighted CSF. This rule is inspired by the deletion-contraction rule for the chromatic polynomial, which is a primary tool for proving theorems. For the traditional CSF, such a rule is not possible as any contraction of an edge necessarily changes the degree of the symmetric function. We state this rule next.

Lemma 1.3.4 ([4])**.** *Let $G = (V, E, \omega)$ be a vertex weighted graph and let $e = (u, v) \in E$ be any edge. Then,*

$$X_{(G,\omega)} = X_{(G \backslash e,\omega)} - X_{(G/e,\omega/e)}$$

where $G \backslash e$ is the graph G with edge e removed, G/e is the graph with the vertices u and v contracted to the vertex w, and $(\omega/e)|_{V-\{u,v\}} \equiv \omega|_{V-\{u,v\}}$ with $(\omega/e)(w) = \omega(u) + \omega(v)$.

Proof. See **Lemma 2** in [4], pg. 6. □

We are now ready to present the proof of a generalization of **Lemma 1.3.3**.

Lemma 1.3.5 ([4])**.** *Let $G = (V, E, \omega)$ be a vertex weighted graph, then,*

$$X_{(G,\omega)} = \sum_{S \subset E} (-1)^{\#S} p_{\lambda(\omega,S)}$$

where $\lambda(\omega, S)$ is the partition of $n = \sum_{v \in V} \omega(v)$ whose components are the total weight of the vertices in each connected component of $G(S)$.

Proof. Let $E = \{e_1, e_2, \dots, e_m\}$. We first apply the deletion-contraction rule (**Theorem 1.3.4**) to the edge e_1, namely,

$$X_{(G,\omega)} = X_{(G\backslash e_1,\omega)} - X_{(G/e_1,\omega/e_1)}$$

Next, we continue this decomposition by applying deletion-contraction to the edge e_2 in the graphs $(G\backslash e_1, \omega)$ and $(G/e_1, \omega/e_1)$, obtaining an expansion of $X_{(G,\omega)}$ with four terms. Similarly, in step i, we apply deletion-contraction to the edge e_i all 2^{i-1} terms in the expansion of $X_{(G,\omega)}$ until we reach step m, at which point we have the equation,

$$X_{(G,\omega)} = \sum_{S\subset E} (-1)^{\#S} X_{(G_S,\omega_S)} \tag{1.3}$$

where (G_S, ω_S) is the graph G with the edges S contracted and deleting all the edges in $E\backslash S$ in order of $e_1, e_2, \dots, e_m$. It is clear that the graph (G_S, ω_S) has no edges, and each vertex corresponds to one connected component in $G(S)$, since they are formed by the contraction of the vertices in each connected components of $G(S)$. Furthermore, the weight of each vertex in (G_S, ω_S) is the sum of the weights of the vertices in a unique connected component of $G(S)$, by our definition of edge contraction on the weight function ω. Lastly, we note that if K_λ the graph with no edges and vertices of weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, then $X_{K_\lambda} = p_\lambda$. Then, the result follows immediately from (1.3). □

Lemma 1.3.5 is indeed a true generalization of **Lemma 1.3.3**, as the latter result can be recovered simply by letting $\omega \equiv 1$.

Lastly, we give a final characterization of the chromatic symmetric function in the power sum basis by considering the *lattice of contractions* $\mathscr{L}_G$ of a graph G, namely, the partially ordered set whose elements are the connected partitions of vertices in G, and whose connections are an ordering by refinement. First, we introduce a beautiful theorem in the study of posets called the *Möbius inversion formula*, which generalizes the inclusion-exclusion principle for counting elements of set unions.

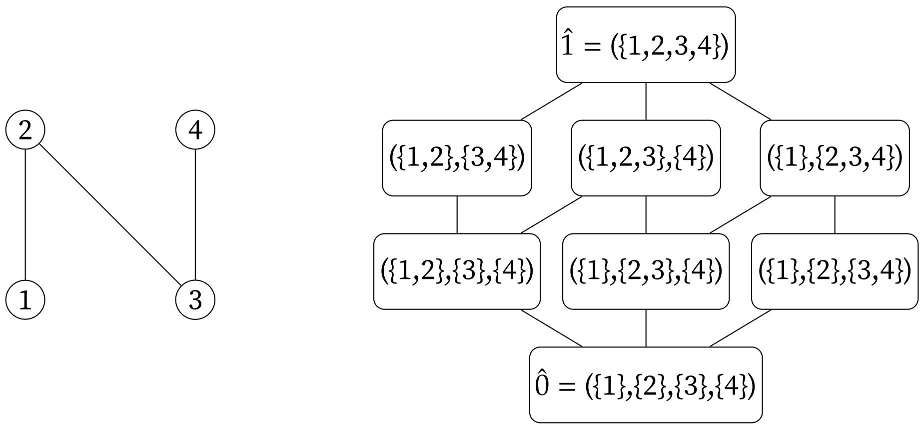


Figure 1.5: A graph (left) and the Hasse diagram of its lattice of contractions $\mathscr{L}_G$ (right).

Lemma 1.3.6 (Möbius inversion formula, [15]). *Let P be a poset such that if $t \in P$, then the set $\Omega_t = \{y \in P : y \le t\}$ has finite cardinality. Let $f, g : P \to K$, where K is a field. Then,*

$$g(t) = \sum_{s\in\Omega_t} f(s), \textit{ for all } t \in P$$

if and only if

$$f(t) = \sum_{s\in\Omega_t} g(s)\mu(s,t), \textit{ for all } t \in P$$

where $\mu : P \to \mathbb{Z}$ *is the Möbius (counting) function, defined by,*

$$\begin{cases} \mu(s,s) = 1 & \textit{for all } s \in P \\ \mu(s,u) = -\sum_{s\leq t<u} \mu(s,t) & \textit{for all } s < u \textit{ in } P \end{cases}$$

Now, we directly get the following result for the CSF.

Lemma 1.3.7.

$$X_G = \sum_{\pi\in\mathscr{L}_G} \mu(\hat{0},\pi) p_{type(\pi)}$$

where $\hat{0}$ *is the minimal element in* $\mathscr{L}_G$*, namely, the set* $\{\{v\} : v \in V(G)\}$*, and* $type(\pi)$ *where* $\pi = (V_1, V_2, ..., V_k) \in \mathscr{L}_G$ *is the partition of* $|V|$ *obtained by sorting* $(|V_1|, |V_2|, ..., |V_k|)$ *in weakly decreasing order. Moreover,* $\mu(\hat{0},\pi)$ *is nonzero for every* $\pi \in \mathscr{L}_G$*.*

Proof. For each $\sigma \in \mathscr{L}_G$, define,

$$X_\sigma = \sum_\kappa x^\kappa$$

where the sum is over all colorings $\kappa : V \to \mathbb{N}$ which give the same color to all vertices in the same block of σ and if $(u,v) \in E$, then $\kappa(u) \neq \kappa(v)$. Then, for each coloring κ_0 of G, there is a unique $\sigma_0 \in \mathscr{L}_G$ such that κ is included in the sum in the definition of X_{σ_0}. It follows that for any $\pi \in \mathscr{L}_G$,

$$p_{\text{type}(\pi)} = \sum_{\sigma\geq\pi} X_\sigma$$

and by the Möbius inversion formula (**Lemma 1.3.6**),

$$X_\pi = \sum_{\sigma\geq\pi} p_{\text{type}(\sigma)} \mu(\pi,\sigma)$$

Finally, the proof follows by noticing that $X_{\hat{0}} = X_G$, where the RHS is the chromatic symmetric function of G. □

1.3.2 Chromatic Bases

One interesting result about the chromatic symmetric function that has inspired much recent work is forming a CSF basis for the set of symmetric functions. Surprisingly, we have the following lemma of Cho and Willigenburg.

Lemma 1.3.8 ([2])**.** *Let* $\{G_k\}_{k\geq1}$ *be a set of connected graphs such that* G_k *has* k *vertices for each* $k \geq 1$*. Moreover, define* $G_\lambda = G_{\lambda_1} + G_{\lambda_2} + ... + G_{\lambda_{l(\lambda)}}$ *where the "+" operation on graphs is as it is in* ***Lemma 1.3.1****, then* $\{X_{G_\lambda} : \lambda \vdash n\}$ *is a* $\mathbb{Q}$*-basis for* Λ^n*.*

Proof. Since the set $\{p_\lambda : \lambda \vdash n\}$ is a $\mathbb{Q}$-basis for Λ^n, it suffices to show that if $X_{G_\lambda} = \sum_{\alpha \vdash n} c_\alpha p_\alpha$, then $c_\lambda \neq 0$. However, it is clear that there exists an element $\pi \in \mathscr{L}_{G_\lambda}$, namely the maximal element $\hat{1}$ such that $\text{type}(\pi) = \lambda$, hence by **Lemma 1.3.7**, $c_\lambda = \mu(\hat{0}, \pi) \neq 0$. Moreover, since $\{X_{G_\lambda} : \lambda \vdash n\}$ contains at most $p(n)$ unique elements and $\dim(\Lambda^n) = p(n)$, it must be a $\mathbb{Q}$-basis for Λ^n. □

Furthermore, the set $\{X_{G_k}\}_{k \geq 1}$ is algebraically independent, like the set $\{e_k\}_{k \geq 1}$, such that one may even consider calling the above lemma the *Chromatic Version* of the Fundamental Theorem of Symmetric Functions. Previous work such as [2, 6] has examined the bases formed by CSFs of families of graphs with simple structures, such as paths and stars. In particular, the main contribution of [2] was giving explicit expansions of several simple families of graphs in known symmetric function bases in order to better understand chromatic bases. Their results are summarized in the following lemma.

Lemma 1.3.9 ([2]). *Let K_n be the complete graph, S_n be the star graph, P_n be the path graph, and C_n be the cycle graph, all on n vertices. Then,*

(i) $X_{K_n} = n! e_n$

(ii) $$X_{S_{n+1}} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} p_{(r+1, 1^{n-r})}$$

(iii) $$X_{P_n} = \sum_{\lambda = (n^{r_n}, \ldots, 1^{r_1}) \vdash n} (-1)^{n - \sum_{i=1}^n r_i} \frac{(\sum_{i=1}^n r_i)!}{\prod_{i=1}^n (r_i)!} p_\lambda$$

(iv) $$X_{C_n} = \sum_{\lambda = (n^{r_n}, \ldots, 1^{r_1}) \vdash n} (-1)^{n - \sum_{i=1}^n r_i} \frac{(\sum_{i=1}^n r_i)!}{\prod_{i=1}^n (r_i)!} \left(1 + \sum_{j=2}^{n} (j-1) \frac{r_j}{\sum_{i=1}^n r_i}\right) p_\lambda + (-1)^n p_n$$

Proof. See **Theorem 8** in [2], pg. 4. □

We extend the results of [2] by giving explicit formulas for the chromatic symmetric function of the complete bipartitie graph $K_{n,m}$ and windmill graph $W_{k,r}$ in the monomial symmetric function basis, and the lollipop graph $L_{n,c}$ in the power sum basis. The complete bipartite graphs $K_{n,m}$ are one of the most famously-studied families of graphs, which consist of a sets of n independent vertices and m independent vertices. Then, every possible edge between the two sets is in the edge set of $K_{n,m}$. The windmill graphs are a generalization of friendship graphs. Specifically, $W_{k,r}$ is formed by taking r independent copies of the complete graph K_k, and adjoining the graphs together at one vertex. See Figure 1.6 for an example of a complete bipartite and windmill graph.

Theorem 1.3.1. *Let $K_{n,m}$ be the complete bipartite graph with $n+m$ vertices and $W_{k,r}$ be the windmill graph which is the composition of r copies of K_k. Then,*

(i) $$X_{K_{n,m}} = \sum_{\lambda \vdash (n+m)} \sum_{\substack{\mu \vdash n \\ \mu \subset \lambda}} \frac{\tilde{\lambda}}{\tilde{\mu} \cdot \widetilde{(\lambda - \mu)}} \frac{n! \cdot m!}{\lambda_1! \lambda_2! \ldots \lambda_{l(\lambda)}!} m_\lambda$$

(ii) $$X_{W_{k,r}} = (k-1)!^r \sum_{\lambda \vdash r(k-1)+1} r_1 M_{(\lambda - (1)), ((k-1)^r)} m_\lambda$$

where r_1 is the number of 1's in λ, and $M_{(\mu, ((k-1)^r)}$ is the number of $l(\mu) \times r$ matrices with entires in $\{0, 1\}$ such that there are exactly $(k-1)$ 1s in each column and μ_j 1s in the j^{th} row.

Proof. (i) By **Lemma 1.3.2**, for each partition $\lambda \vdash n+m$ the coefficient of $\widetilde{m}_\lambda$ in $X_{K_{n,m}}$ is the number of partitions of V into independent sets of type λ. Let V_1 be the set of n vertices in $K_{n,m}$ with no edges between them. Likewise, let V_2 be the set of m vertices in $K_{n,m}$ with no edges between then. Since for every $u \in V_1$ and $v \in V_2$, $(u,v) \in E(K_{n,m})$, then any independent set in $K_{n,m}$ must be a subset of either V_1 or V_2. We first pick the sizes of the independent sets in V_1, represented by $\mu = (\mu_1, ..., \mu_k) \subset \lambda$. Suppose that we have not chosen any independent sets from V_1, then there are $\binom{n}{\mu_1}$ ways to choose an independent set of size μ_1, since each vertex in V_1 in independent with respect to every other vertex in V_1. Likewise, there are $\binom{n-\mu_1}{\mu_2}$ ways to choose the independent set of size μ_2 from the remaining $n-\mu_1$ vertices in V_1 after the first independent set has already been chosen. Continuing in this way, there are,

$$\prod_{j_1}^{l(\mu)} \binom{n - \sum_{\ell=1}^{j_1-1} \mu_\ell}{\mu_{j_1+1}} = \frac{n!}{\mu_1!(n-\mu_1)!} \frac{(n-\mu_1)!}{\mu_2!(n-\mu_1-\mu_2)!} \cdots \frac{(n-\sum_{\ell=1}^{j_1-1}\mu_\ell)!}{\mu_{l(\mu)}!(n-\sum_{\ell=1}^{l(\mu)}\mu_\ell)!} = \frac{n!}{\mu_1!\mu_2!...\mu_{l(\mu)}!}$$

ways to choose independent sets with sizes corresponding to μ from the set V_1 where each independent set is distinct.[5] It is not hard to see that, by the same logic, there are,

$$\prod_{j_2=1}^{l(\lambda)-l(\mu)} \binom{m - \sum_{\ell=1}^{j_2-1}(\lambda-\mu)_\ell}{(\lambda-\mu)_{j_2+1}} = \frac{m!}{(\lambda-\mu)_1!(\lambda-\mu)_2!...(\lambda-\mu)_{l((\lambda-\mu))}!}$$

ways to choose independent sets with sizes corresponding to $\lambda-\mu$ from V_2 where each independent set is distinct. Lastly, we must account for the overcounting in the choice of independent sets. Namely, if there are multiplicities in the partition μ or $(\lambda-\mu)$, then we are overcounting the choices of independent sets with the size of these repeated numbers as the order in which they are selected does not matter. To account for this, we divide each product in the sum by $1/\tilde{\mu}$ and $1/\widetilde{(\lambda-\mu)}$, respectively.

(ii) We first note that there are no nontrivial independent sets (those containing more than one vertex) containing the center vertex (where all of the complete graphs are adjoined) in $W_{k,r}$, since it is connected to every other vertex in $W_{k,r}$. Hence, for any partition $\lambda = ((\#V)^{r_{\#V}}, ..., 1^{r_1}) \vdash \#V$ such that $r_1 = 0$, we must have that the coefficient of m_λ in $X_{W_{k,r}}$ is. 0. Therefore, we restrict ourselves to partitions with $r_1 > 0$.

Assuming that the central vertex is in an independent set of size 1, we wish to find the number of ways to groups the remaining vertices, which can be thought-of as r independent copies of K_{k-1} into independent sets of size $\lambda - (1)$. We call the graph of $W_{k,r}$ with the central vertex and all of its adjacent edges removed $W'_{k,r}$. Moreover, by (i) in **Lemma 1.3.9**,

$$\begin{aligned} X_{W'_{k,r}} &= (X_{K_{k-1}})^r \\ &= ((k-1)!e_{k-1})^r \\ &= (k-1)!^r e_{((k-1)^r)} \\ &= (k-1)!^r \sum_{\lambda \vdash r(k-1)} M_{\lambda,((k-1)^r)} m_\lambda \end{aligned}$$

[5] This is equivalent to the counting performed by "multinomial coefficients", which are not used here for improved clarity of the result.

where we used **Lemma 1.3.1** in the first step and **Lemma 1.2.2** in the final step. Hence, by the interpretation of the augmented monomial symmetric functions from **Lemma 1.3.2**, there are, $(1/\widetilde{\lambda - (1)})(k-1)!^r M_{(\lambda-(1)),((k-1)!^r)}$ ways to choose independent sets of size $\lambda - (1)$ from $W'_{k,r}$. Therefore the coefficient of $\widetilde{m}_\lambda$ in $X_{W_{k,r}}$ is $(1/\widetilde{\lambda - (1)})(k-1)!^r M_{(\lambda-(1)),((k-1)!^r)}$, or, equivalently, the coefficient of m_λ in $X_{W_{k,r}}$ is $(\widetilde{\lambda}/\widetilde{\lambda - (1)})(k-1)!^r M_{(\lambda-(1)),((k-1)!^r)} = r_1(k-1)!^r M_{(\lambda-(1)),((k-1)!^r)}$. Conveniently, this selects all partitions such that $r_1 > 0$, so we don't need any restrictions in the sum over λ. □

Example 1.3.1. We verify the formula (ii) from **Theorem 1.3.1** for the simple case where $W_{k,r} = (V, E)$ and $\lambda = (1^{\#V})$. Since there is only one way to partition the vertex set into independent sets each of size 1, namely, by putting each vertex into its own independent set, **Lemma 1.3.2** tells us that the coefficient of $\widetilde{m_\lambda}$ in $X_{W_{k,r}}$, denoted by $[\widetilde{m_\lambda}]_{W_{k,r}}$, should be exactly 1. That is, $[m_\lambda]_{W_{k,r}} = (\#V)! = (r(k-1)+1)!$.

We first compute the value of $M_{(1)^{r(k-1)},((k-1)^r)}$, which is the number of $r(k-1) \times r$ 0,1-matrices such that there is a single 1 in each row and $k-1$ 1s in each column. Every such matrix is a permutation of the rows of the matrix A which has $A_{i,j} = 1$ where $(i-1)(k-1) < j \leq i(k-1)$ and $A_{i,j} = 0$ otherwise. There are $(r(k-1))!$ total such permutations. Moreover, for any $(i-1)(k-1) < j_1 < j_2 \leq i(k-1)$, permuting rows j_1 and j_2 in A does not change the matrix (since the rows are identical). There are $(k-1)!^r$ total such permutations. It follows that $M_{(1)^{r(k-1)},((k-1)^r)} = r(k-1)!/(k-1)!^r$.

Lastly, we note that $r_1 = r(k-1)+1$ where $\lambda = ((\#V)^{r_{\#V}}, \ldots, 1^{r_1})$. Finally, (ii) in **Theorem 1.3.1** gives the following expression.

$$[m_\lambda]_{W_{k,r}} = (k-1)!^r \cdot (r(k-1)+1)\frac{(r(k-1))!}{(k-1)!^r} = (r(k-1)+1)!$$

□

Interestingly, equation (ii) from **Theorem 1.3.1** gives a correspondence between the numbers $M_{\lambda,\mu}$ under certain conditions and the number of independent sets in a graph. In our proof, we relied on an algebraic abstraction to the elementary symmetric functions, however, one wonders if the same result can be shown in a purely combinatorial way, as there is a simple combinatorial interpretation of the coefficients $M_{\lambda,\mu}$. It's worth nothing that the coefficients $M_{\lambda,\mu}$ are related to the Kostka numbers, which express the decomposition of permutation modules in terms of the irreducible representations of the symmetric group, and hence play a key role in the field of algebraic combinatorics.

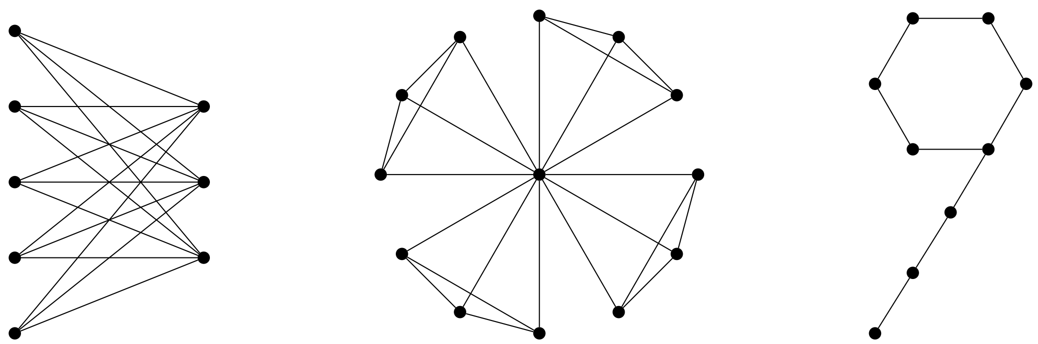


Figure 1.6: Complete bipartite graph $K_{5,3}$ (left), windmill graph $W_{4,4}$ (center), and lollipop graph $L_{9,6}$ (right).

A *lollipop graph* on n vertices with girth $c < n$ is the cycle graph C_c with a path of length $n - c$ attached to any vertex in the cycle. See Figure 1.6 for a representation of $L_{9,6}$. The next theorem gives an explicit formula for the chromatic symmetric function of any, arbitrary lollipop graph. In the statement of the next theorem, we abuse the notation of the so-called "Kronecker delta" by defining for a condition S and an object O,[6]

$$\delta_S(O) = \begin{cases} 1 & \text{if } O \text{ satisfies } S \\ 0 & \text{if } O \text{ does not satisfy } S \end{cases}$$

Theorem 1.3.2. *Let $L_{n,c}$ be the unique lollipop graph on n vertices with girth c, then,*

$$X_{L_{n,c}} = \sum_{\lambda \vdash n} C_\lambda^{n,c} p_\lambda$$

where $\lambda = (n^{r_n}, \ldots, 1^{r_1})$,

$$C_\lambda^{n,c} = C_{\lambda-(1)}^{n-1,c} \cdot \delta_{\{r_1>0\}}(\lambda) + \sum_{\{1<k\leq n-c:r_k>0\}} C_{\lambda-(k)}^{n-k,c}$$
$$+ \sum_{\{n-c<k\leq n:r_k>0\}} (k+c-n) \cdot (-1)^{(n-k)-(\sum_{i=1}^n r_i - 1)} \frac{(\sum_{i=1}^n r_i - 1)! \cdot r_k}{\prod_{i=1}^n (r_i)!} + (-1)^n \cdot \delta_{\{r_n=1\}}(\lambda)$$

and,

$$C_\lambda^{c+1,c} = (-1)^{c-\sum_{i=1}^{c+1} r_i - 1} \frac{(\sum_{i=1}^{c+1} r_i - 1)! \cdot r_1}{\prod_{i=1}^{c+1} (r_i)!} \left(1 + \sum_{j=2}^{c} (j-1) \frac{r_j}{\sum_{i=1}^{c+1} r_i - 1}\right) \cdot \delta_{\{r_1>0\}}(\lambda) + (-1)^c \delta_{\{r_c=1\}}(\lambda)$$
$$+ \sum_{\{1<k\leq c+1:r_k>0\}} (k-1) \cdot (-1)^{(n-k)-(\sum_{i=1}^{c+1} r_i - 1)} \frac{(\sum_{i=1}^{c+1} r_i - 1)! \cdot r_k}{\prod_{i=1}^{c+1} (r_i)!} + (-1)^{c+1} \cdot \delta_{\{r_{c+1}=1\}}(\lambda)$$

Proof. We proceed by induction on the length of the path adjoined to the cycle in $L_{n,c}$, that is, on n. For the base case ($n = c + 1$), we attach a leaf arbitrarily to a vertex in the cycle graph with c vertices to form $L_{c+1,c}$. Let $e \in E(L_{c+1,c})$[7] be the unique leaf edge in $L_{c+1,c}$. For a fixed $\lambda_0 \vdash n$, we wish to apply **Lemma 1.3.3** by counting the number of subsets $S \subset E(L_{c+1,c})$ such that $\lambda_{L_{n,c}}(S) = \lambda_0$. We divide our analysis into three disjoint cases.

In the first case, we have $e \notin S$, in which case the unique leaf of $L_{c+1,c}$ will always be disconnected from the rest of the graph. Hence, if $r_1 = 0$ in λ_0, there is no way to choose a subset S such that $\lambda_{L_{n,c}}(S) = \lambda_0$. Assume, then, that $r_1 > 0$, then the number of ways to choose S such that $\lambda_{L_{n,c}}(S) = \lambda_0$ is exactly the number of ways to choose a subset $S_0 \subset E(C_c)$ where C_c is the cycle graph on c vertices, such that $\lambda_{C_c}(S_0) = \lambda_0 - (1)$. Part (iv) of **Lemma 1.3.9** completes the argument, giving us the term,

$$(-1)^{c-\sum_{i=1}^{c+1} r_i - 1} \frac{(\sum_{i=1}^{c+1} r_i - 1)! \cdot r_1}{\prod_{i=1}^{c+1} (r_i)!} \left(1 + \sum_{j=2}^{c} (j-1) \frac{r_j}{\sum_{i=1}^{c+1} r_i - 1}\right) \cdot \delta_{\{r_1>0\}}(\lambda) + (-1)^c \delta_{\{r_c=1\}}(\lambda)$$

[6] I credit Dr. Pascal Grange for introducing this notation to me.

[7] For a graph $G = (V, E)$, we let $E(G) = E$, that is, $E(G)$ is the edge set of G. Similarly, we say $V(G) = V$ is the vertex set of G.

In the second case, $e \in S$ and $S \neq E(L_{c+1,c})$. In this case, we suppose that e is in a connected component of $L_{c+1,c}(S)$ with vertex size k, where $k > 1$ is fixed. The key observation of this proof is that **the remaining graph without the connected component containing e is a path** (see Figure 1.7), so the number of ways to choose S given e is in a connected component of size k is the number of ways to choose a subset $S_1 \subset E(P_{n-k})$ where P_{n-k} is the path graph on $n-k$ vertices such that $\lambda_{P_{n-k}}(S_1) = \lambda - (k)$. Moreover, there are $k-1$ ways to position the connected component containing e on the graph of $L_{c+1,c}$. Part (iii) of **Lemma 1.3.9** completes the argument, giving us the term,

$$\sum_{\{1<k\leq c+1: r_k>0\}} (k-1)\cdot(-1)^{(n-k)-(\sum_{i=1}^{c+1} r_i - 1)} \frac{(\sum_{i=1}^{c+1} r_i - 1)! \cdot r_k}{\prod_{i=1}^{c+1}(r_i)!}$$

In the last case, $S = E$, from which we get the term $(-1)^{c+1}\delta_{\{r_{c+1}=1\}}(\lambda)$.

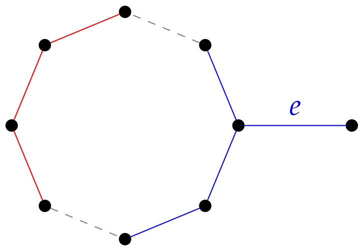


Figure 1.7: Fixing the size of the connected component containing e in $L_{9,8}$ to be 5, the remaining graph is always a path on 4 vertices, that is P_4. Moreover, there are 4 ways to position the blue edges.

For the inductive step, suppose that for any $\lambda_0 \vdash b$, $C^{b,c}_{\lambda_0}$ correctly counts the number of ways to choose a subset $S \subset E(L_{b,c})$ such that $\lambda_{L_{n,c}}(S) = \lambda_0$, and $1 \leq b < n$. Let e be the unique leaf edge in $L_{n,c}$. In the first case, we suppose that $e \notin S$, in which case there are exactly $C^{n-1,c}_{\lambda-(1)}$ ways to choose a subset $S \subset E(L_{n,c})$ with $\lambda(S) = \lambda$ if $r_1 > 0$, and 0 otherwise.

In the second case, $e \in S$, and we consider the size of the connected component of $L_{n,c}(S)$ which contains e. Suppose the vertex size of this connected component is $1 < k \leq n-c$, then the remaining graph besides the component containing e is exactly $L_{n-k,c}$, from which we conclude there are $C^{n-k,c}_{\lambda-(k)}$ ways to choose a subset S with $\lambda_{L_{n,c}}(S) = \lambda$. If the size of the connected component containing e is greater than $n-c$, then the analysis is the same as in the base case, namely, the remaining graph of $L_{n,c}(S)$ except for the connected component containing e is a path graph. In this case, there are notably $k-(n-c)$ ways to place the connected component containing e onto the graph of $L_{n,c}$. The result follows from **Lemma 1.3.9**. □

We end with a short discussion about chromatic bases. Recently, it has been shown in [6] that there is a simple algorithm for computing the CSF of a tree in the star basis. Moreover, by determining the smallest partition of n in lexicographic order that has a nonzero coefficient in the star-basis expansion of the CSF, they show that all trees of diameter less than 5 are distinguished by the CSF. Furthermore, they prove that the set of chromatic symmetric functions of all trees is a $p(n)-n+1$ dimension subspace of the set of symmetric functions. It remains to be shown what other nontrivial properties can be proven by examining the CSF in other chromatic bases, such as the path basis.

Moreover, another question about chromatic bases is which of these bases are *Schur positive*, namely, each basis element is a positive linear combination of Schur symmetric functions. For example, the basis $\{X_{K_n} : n \geq 1\}$ is Schur positive as we have the relationship $e_n = s_{(1^n)}$.

Chapter 2

GRAPH PROPERTIES PRESERVED BY THE CSF

Stanley [14] gave a notable example of two nonisomorphic graphs on 5 vertices which share the same chromatic symmetric function. This example negatively answers the question of whether the chromatic symmetric function distinguishes all nonisomorphic graphs. However, the question still remains open, and is conjectured to be true, for tree graphs (see **Chapter 3**). A natural extension of Stanley's counterexample is to obtain a complete classification of the graph properties preserved by the chromatic symmetric function. In this section, we give a review of the major graph and tree properties encoded in the chromatic symmetric function known so-far. The proofs for **Lemmas 2.1.1, 2.1.2, 2.1.3, 2.2.1** were discovered independent of a reference for this paper, although the results themselves are well-known.

Figure 2.1: Stanley's counterexample of two non-isomorphic graphs on 5 vertices with the same CSF. The coloring of each graph is proper and corresponds to the term ${x_1}^2{x_2}^2{x_3}^1$ in their CSF.

To review some notation from the previous section, we let $[p_\lambda]_G$ (or any other base, for that matter) be the coefficient of p_λ in X_G.

2.1 Graph Properties Preserved by the CSF

2.1.1 Number of Vertices and Edges

It is not hard to see that the chromatic symmetric function preserves the number of edges and vertices of the graphs to which it corresponds, which is summarized in the following lemma.

Lemma 2.1.1. *Let $G = (V, E)$ be a graph. X_G preserves the number of vertices $n = \#V$ and the number of edges $m = \#E$ in G.*

Proof. By definition, $X_G = \sum_\kappa \prod_{v\in V} x_{\kappa(v)}$ where $\kappa : V \to \mathbb{N}$ is a proper coloring of G. Hence, each monomial in X_G has degree n.

For the number of edges, we note that for an edge subset $S \subseteq E$, $\lambda(S) = (2, 1^{n-2})$ if and only if $\#S = 1$. It follows immediately from **Lemma 1.3.3** that the number of edges $m = [p_{(2,1^{n-2})}]_G$. □

2.1.2 Number and Size of Connected Components

Like the number of vertices and edges, the number of connected components of a graph is immediate from the combinatorial interpretation of the CSF in the power sum basis.

Lemma 2.1.2. *X_G preserves the size and number of connected components in $G = (V, E)$.*

Proof. Let $X_G = \sum_{\lambda \vdash n} c_\lambda p_\lambda$, and let $\lambda_0 = (\lambda_1, \lambda_2, ..., \lambda_{l(\lambda)})$ be the largest partition of n in lexicographic order such that $c_{\lambda_0} \neq 0$. By **Lemma 1.3.3**, $c_\lambda \neq 0$ if and only if there is some subset of edges $S \subseteq E$ such that the connected components of $G(S)$ have size $\lambda_1, \lambda_2, ..., \lambda_{l(\lambda)}$, accordingly. Furthermore, removing edges from G can only have the effect of increasing the number of connected components. It follows that $\lambda(S) = \lambda_0$ if and only if $S = E$. Hence, $l(\lambda_0)$ is the number of connected components in G and $(\lambda_1, \lambda_2, ..., \lambda_{l(\lambda)})$ is the sequence of the sizes of connected components in G. □

2.1.3 Number of *k*-Edge Matchings

For $k \in \mathbb{Z}^+$, a k-edge matching is a subset of edges $S \subseteq E$ such that $\#S = k$ and each edge in S is disjoint from every other edge in S. In other words, each edge in S connects two *unique* vertices in G.

Lemma 2.1.3. *Let $k \in \mathbb{Z}^+$. X_G preserves the number of k-edge matchings of G.*

Proof. We will once again consider the power sum symmetric function basis. The number of k-edge matchings is simply the number of ways to choose $S \subseteq E$ such that there are exactly k connected components of size 2 in $G(S)$ and $n-2k$ connected components of size 1. By **Lemma 1.3.3**, this number is $[p_{(2^k, 1^{n-2k})}]$ in the power sum expansion of X_G. □

2.1.4 Number of Triangles

A graph *triangle* is defined as K_3, or the complete graph of 3 vertices. The number of triangles in a graph G is the number of induced subgraphs of K_3 that exist in G. For example, the complete graph K_n has $\binom{n}{3}$ triangles (one for each triplet of vertices). The following clever argument from Orellana and Scott [13] establishes that the chromatic symmetric function captures the number of triangles in a graph.

Lemma 2.1.4 ([13]). *X_G preserves the number of triangles, T_G, in $G = (V, E)$.*

Proof. Let $S^{(2,2)}$ be the number of 2-edge matchings in G (see **Subsection 2.1.3**). By **Lemma 2.1.3**, $S^{(2,2)} = [p_{(2^2, 1^{n-2^2})}]$ in X_G. Furthermore, let $S^{(3)}$ be the number of edge subsets of size 2 such that $G(S)$ contains three connected vertices. We have that $\binom{\#E}{2} = S^{(2,2)} + S^{(3)}$. By **Lemma 2.1.1**, X_G preserves $\#E$. It follows directly that X_G preserves $S^{(3)}$. Finally, by **Lemma 1.3.3**, $[p_{3,1^{(n-3)}}] = S^{(3)} + T_G$, so X_G preserves the number of triangles in G. □

2.1.5 Sum of the Vertex Degrees Squared

In general, the chromatic symmetric function cannot distinguish the degree sequence of graphs, as there are pairs of nonisomorphic graphs known for which their chromatic symmetric function is the same, yet they have differing degree sequences (such as Stanley's counterexample in Figure 3.2). However, for the special case of trees, the degree sequence is known to be preserved by the CSF (see **Section 2.2.2**). A slightly weaker result is known that the sum of the squares of the degrees in general graphs are preserved by their CSFs. This result was proven originally by Orellana and Scott [13].

Lemma 2.1.5 ([13]). *X_G preserves the sum of the squared vertex degrees of a graph $G = (V_1, E_1)$, namely the quantity,*

$$\sum_{v \in V(G)} d(v)^2$$

Proof. Let $S^{(2,2)}$ and $S^{(3)}$ be defined as in the proof of **Lemma 2.1.4.** In particular, $\binom{\#E}{2} = S^{(2,2)} + S^{(3)}$, so by **Lemma 2.1.1**, X_G preserves $S^{(3)}$. Suppose that we are given another graph $H = (V_2, E_2)$ such that $X_G = X_H$. We can count $S^{(3)}$ by noting that it is uniquely determined by the choice of a central vertex and two vertices incident to the central vertex. In specific, since $X_G = X_H$, the number of ways to pick a central vertex and then pick two edges incident to that vertex is the same in both G and H. In equations,

$$\sum_{v \in V_1} \binom{d_G(v)}{2} = \sum_{v \in V_2} \binom{d_H(v)}{2} \implies \sum_{v \in V_1} (d_G(v)^2 - d_G(v)) = \sum_{v \in V_2} (d_H(v)^2 - d_H(v))$$
$$\implies \sum_{v \in V_1} d_G(v)^2 = \sum_{v \in V_2} d_H(v)^2$$

where the last implication follows from **Lemma 2.1.1** and the fact that, for any graph, $\sum_{v \in V} d(v) = 2 \cdot \#E$. ☐

2.1.6 Girth

Girth is one of the main graph invariants used to classify and study graphs. The girth of a graph is the length of the shortest cycle within a graph. For example, the cycle graph C_n clearly has girth n for $n \geq 3$. More interestingly, the famous *Petersen* graph has a girth of 5. In 2008, Martin et al. [12] showed that the chromatic symmetric function of a graph preserves its girth, which is summarized in the following lemma.

Lemma 2.1.6 ([12]). *X_G preserves the girth g of G.*

Proof. To begin, let $k \in \mathbb{Z}^+$ such that $k > n - g + 1$. It follows that $n - k < g - 1$. We will show that for a subset $S \subset E$, $l(\lambda(S)) = k$ if and only if $\#S = n - k$. Firstly, since $n - k < g$, any such subset S must be acyclic, and therefore $G(S)$ has exactly k connected components.[1] Alternately, suppose that $G(S)$ has exactly k connected components, then $l(\lambda(S)) = k$. The maximum size of any component of $G(S)$ is $n - (k - 1) < g$, so $G(S)$ must be acyclic, therefore it has $n - k$ edges. **Lemma 1.3.3** implies that,

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda) = k}} [p_\lambda]_G = \sum_{\substack{S \subseteq E \\ \lambda(S) = \lambda}} (-1)^{\#S} = \sum_{\substack{S \subseteq E \\ \#S = n-k}} (-1)^{\#S} = (-1)^{n-k} \cdot \binom{\#E}{n-k}$$

[1] It is not hard to show this fact by induction. Namely, suppose G is an acyclic graph with $n - e$ edges and k connected components. Since G is acyclic, removing an edge must disconnect one connected component into two connected components, giving a graph G' with $n - (e + 1)$ edges and $k + 1$ connected components. Clearly, once G' has n connected components, it is the empty graph on n vertices, so $n - (e + (n - k)) = 0$, hence $e = k$.

Now, suppose that $k = n - g + 1$. We will show that for a subset $S \subseteq E$, $G(S)$ has k connected components if and only if either it (1) has $n - k$ edges or (2) is a cycle of length g. To begin, suppose that $G(S)$ has k connected components. If $\#S = n - k$, we are done. So, suppose that $\#S \neq n - k$, that is, $\#S \geq n - k + 1 = g$, then by definition of the girth g, $G(S)$ must contain a cycle C. However, since $G(S)$ has k connected components, there are at least $k - 1$ vertices that do not belong to C. It follows directly that $g \leq \#V(C) \leq n - (k - 1) = g$, so C is a cycle of length g. Furthermore, S cannot have any edges that do not belong to C, since this would imply that $G(S)$ has fewer than k components.

Going in the other direction, (1) implies that $G(S)$ has k components by the same reasoning as before. Alternately, if $G(S)$ is a cycle of length $g = n - k + 1$, then there are $k - 1$ vertices not in the cycle, hence $G(S)$ has k connected components. Letting Γ denote the set of g cycles in G, we have that,

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda)=k}} [p_\lambda]_G = \sum_{\substack{S \subseteq E \\ \lambda(S)=\lambda}} (-1)^{\#S} + \sum_{C \in \Gamma} (-1)^{\#V(C)} = (-1)^{n-k} \cdot \binom{\#E}{n-k} + (-1)^{n-k+1} \cdot \#\Gamma \neq (-1)^{n-k} \cdot \binom{\#E}{n-k}$$

since $\#\Gamma > 0$. Let k be the largest such that $\sum_{\substack{\lambda \vdash n \\ l(\lambda)=k}} [p_\lambda]_G \neq (-1)^{n-k} \cdot \binom{\#E}{n-k}$. It follows that $g = n - k + 1$. □

2.2 Tree Properties Preserved by the CSF

2.2.1 Number of Subtrees with j Vertices

A subtree is a connected subgraph of a tree. In the power sum symmetric function basis, it is straightforward to see that chromatic symmetric function of a tree captures the number of subtrees with $j \in \{1, \ldots, n\}$ vertices.

Lemma 2.2.1. *Let T be a tree graph. Then X_T preserves the number of subtrees of T with j vertices for $j \in \{1, \ldots, n\}$.*

Proof. Any subgraph of a tree with one connected component of size j and all other connected components of size 1 must be a subtree with j vertices adjoined with $n - j$ independent vertices. Likewise, if $S \subseteq E$ is the edge set of a subtree with j vertices, then $G(S)$ has one connected component of size j and $n - j$ connected components of size 1. It follows from **Lemma 1.3.3** that the number of subtrees with j vertices is $[p_{(j,1^{n-j})}]$. □

2.2.2 Degree and Path Sequence

The *degree sequence* of a graph G is the set of vertex degrees in G sorted in weakly decreasing order. Similarly, the *path sequence* of G is defined as the sequence $P(G) = (P_0(G), P_1(G), \ldots, P_\rho(G))$, where $P_i(G)$ is the number of inducted path subgraphs P_i are contained in G, and ρ is the length of the longest induced path in G.

It was proven by Martin, Morin, and Wagner [12] in 2008 that the chromatic symmetric function determines another graph polynomial, called the *bivariate subtree polynomial* $\mathbf{S}_T$ of a tree T. In specific, $\mathbf{S}_T$ is defined in the following way,

$$\mathbf{S}_T = \mathbf{S}_T(q, r) = \sum_{\text{subtrees } S} q^{\#S} r^{\#L(S)}$$

where $L(S)$ is the set of leaf edges in $T(S)$. Furthermore, it is known that the bivariate subtree polynomial preserves the degree sequence and path sequence of a graph. We direct the reader to [1] for a full proof that $\mathbf{S}_T$ preserves the degree and path sequences of T.

In particular, [12] proved that the bivariate subtree polynomial can be expressed as a linear combination over the partitions $\lambda \vdash n = \#V$ of the number of subsets of the edge set $E(G)$ such that $\lambda(S) = \lambda$. By **Lemma 1.3.3**, this implies that X_T contains enough information to reconstruct $\mathbf{S}_T$, when expressed in the power sum basis. The formal statement of the theorem from [12] is given next.

Lemma 2.2.2 ([12]). *For every $n \geq 1$ and for every tree T with n vertices,*

$$\mathbf{S}_T(q, r) = \sum_{i=1}^{n-1} \sum_{j=1}^{i} q^i r^j \sum_{\lambda \vdash n} \phi(\lambda, i, j) c_\lambda(T)$$

where

$$\phi(\lambda, i, j) = (-1)^{i+j} \binom{l(\lambda) - 1}{l(\lambda) - n + i} \sum_{d=1}^{j} (-1)^d \binom{i-d}{j-d} \sum_{k=1}^{l(\lambda)} \binom{\lambda_k - 1}{d}$$

and $X_T = \sum_{\lambda \vdash n} c_\lambda(T) p_\lambda$.

Proof. See **Theorem 1** in [12], pg. 7. □

Using their result, [12] showed that the chromatic symmetric function distinguishes the infinite family of spider trees. That is, the degree and path sequences of T are sufficient to determine T, if T is a spider.

2.2.3 Trunk Size and Twig Sequence

Crew [3] extended the result of [12] by showing that the information contained in $\mathbf{S}_T$ alonside the known properties which are preserved by X_T are sufficient to recover the size of the *trunk* and the length of the *twigs* in any tree T.

The trunk of a tree T, denoted by T°, is the minimal subtree which contains all the vertices of degree at least 3 in T. By contrast, a twig corresponding to a leaf l of T is the longest path in T containing l such that every vertex which is not an endpoint of the twig has degree 2. We give a rough proof that X_T preserves T°, but refer the reader to [3] for the proof that the length of the twigs are preserved.

Lemma 2.2.3 ([3]). *From X_T, we can recover (i) the size of T° and (ii) the length of all twigs of T.*

Proof. (of (i)) Suppose that T has a leaves. In particular, by **Lemma 2.2.1**, we can recover the value a from X_T. Suppose that S is a subtree of T with a leaves. Then, S must contain the trunk T°. Moreover, S must contain every edge which is adjacent to a vertex of degree greater than 3. Hence, the smallest subtree with a leaves is the subtree which contains the trunk T° and one edge for each of the a leaves in T. From **Lemma 2.2.2**, we can recover $\mathbf{S}_T$ from X_T. From $\mathbf{S}_T$, we find the smallest value v such that there exists a subtree of T with v vertices and a leaves. It follows from the above reasoning that $\#V(T^\circ) = v - a$. The proof of (ii) follows from a simple argument and can be found in [3]. □

Chapter 3

STANLEY'S ISOMORPHISM CONJECTURE

Professor Richard Stanley not only introduced the chromatic symmetric function in 1995, but also stated two big questions, motivating its further study. The first asked for sufficient conditions for the CSF to the e-positive, that is, have positive coefficients in the elementary symmetric function basis, called the "Stanley-Stembridge Conjecture", and has recently been solved [10].[1] The other still remains open to this day, and is commonly dubbed *Stanley's isomorphism conjecture*, *Staney's tree conjecture*, or *Stanley's tree isomorphism conjecture*.

In particular, Stanley proposed that two trees are isomorphic if and only if they have the same CSF. So far, this conjecture is widely believed to be true, and has been verified for trees up to 29 vertices [9]. Proving the conjecture would have applications to many far-reaching areas of mathematics and beyond. For example, it would allow us to efficiently implement algorithms to check for tree isomorphisms by checking any of the characterizations of the coefficients for the CSF of trees in any symmetric function basis, such as the number of independent sets corresponding to each partition of n (see **Lemma 1.3.2**).

In this chapter, we begin in **Section 3.1**, introducing the topic of Stanley's isomorphism conjecture by considering a simple, infinite family of trees which are constructed by adjoining a leaf to a path graph. Then, in **Section 3.2**, we extend the results of [12] and [3] by showing that spider trees can be *reconstructed* using the information in their CSF. This work culminates in **Section 3.3**, where we give a new approach to the proof of Stanley's conjecture based off induction on the "isomorphism distance" between two trees.

3.1 Warm Up: Path Graphs

To begin this section, we give a simple, combinatorial argument to show that graphs produced by adjoining a leaf to a path graph can be distinguished from one another based on their CSF.

In particular, let P_n be the path graph with n vertices, and let $P_{n,k}$ denote P_n with a leaf adjoined to the k^{th} vertex, where vertices are labeled sequentially such that the vertex labeled 1 is a leaf, and $k \in \{1, \dots, \lceil n/2 \rceil\}$. We wish to show that by studying $X_{P_{n,k}}$, we can recover k, and hence reconstruct the graph. We begin by giving a few examples of these types of graphs, and their corresponding CSFs in the power sum basis.

[1] It suffices for G to be the incomparability graph of a "claw free" poset, that is, a $(3 + 1)$-free poset.

$$\text{(left) } P_{3,1} \xrightarrow{X} -p_{(4)} + 2p_{(3,1)} + p_{(2,2)} - 3p_{(2,1,1)} + p_{(1,1,1,1)}$$

$$\text{(right) } P_{3,2} \xrightarrow{X} -p_{(4)} + 3p_{(3,1)} + 0p_{(2,2)} - 3p_{(2,1,1)} + p_{(1,1,1,1)}$$

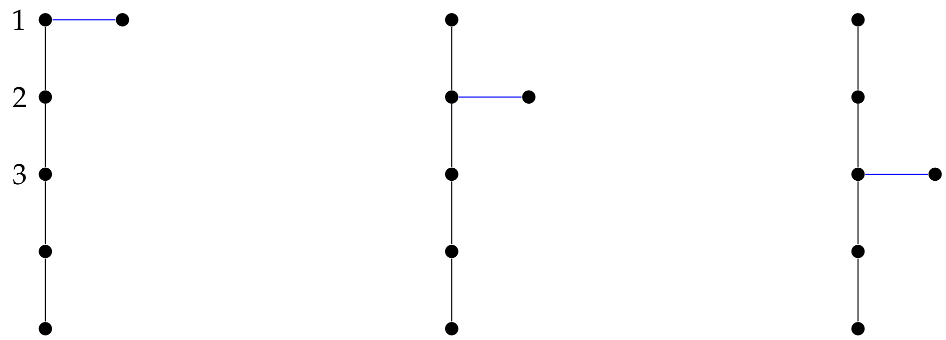
We know that, if Stanley's conjecture is true, the *information* of the structural difference in the two graphs must be contained in the algebraic difference of these symmetric functions, namely, it is encoded in the object,

$$X_{P_{3,1}} - X_{P_{3,2}} = -p_{(3,1)} + p_{(2,2)}$$

We continue to look at more examples to try and make a conjecture about the way this information is encoded. For convenience, we define the notation $o(\lambda)$ as set of partitions $\mu \vdash n$ for which $\mu <_L \lambda$ where $<_L$ is the lexicographic (total) ordering on partitions. Likewise, we let $o(p_\lambda) = \sum_{\mu \in o(\lambda)} a_\mu p_\mu$ for some undetermined coefficients a_μ which may be 0.

$$\text{(left) } P_{4,1} \xrightarrow{X} p_5 - 2p_{(4,1)} - 2p_{(3,2)} + 3p_{(3,1,1)} + o(p_{(3,1,1)})$$

$$\text{(right) } P_{4,2} \xrightarrow{X} p_5 - 3p_{(4,1)} - p_{(3,2)} + 4p_{(3,1,1)} + o(p_{(3,1,1)})$$



$$\text{(left) } P_{5,1} \xrightarrow{X} -p_{(6)} + 2p_{(5,1)} + 2p_{(4,2)} + p_{(3,3)} + o(p_{(3,3)})$$

$$\text{(center) } P_{5,2} \xrightarrow{X} -p_{(6)} + 3p_{(5,1)} + p_{(4,2)} + p_{(3,3)} + o(p_{(3,3)})$$

$$\text{(right) } P_{5,3} \xrightarrow{X} -p_{(6)} + 3p_{(5,1)} + 2p_{(4,2)} + 0p_{(3,3)} + o(p_{(3,3)})$$

For simplicity, we define for a fixed n, $[k:j]_n = X_{P_{n,k}} - X_{P_{n,j}}$. Then, we have,

$n=4$	$n=5$	$n=6$
$[1:2]_3 = -p_{(3,1)} + p_{(2,2)}$	$[1:2]_4 = p_{(4,1)} - p_{(3,2)} + o(p_{(3,1,1)})$	$[1:2]_5 = -p_{(5,1)} + p_{(4,2)} + o(p_{(3,2,1)})$
		$[1:3]_5 = -p_{(5,1)} + p_{(3,3)} + o(p_{(3,2,1)})$
		$[2:3]_5 = -p_{(4,2)} + p_{(3,3)} + o(p_{(3,2,1)})$

It is not difficult to see that for $n = 4, 5, 6$, the two leading terms in lexicographic order of $[k:j]_n$ where $j \neq k$ always are indexed by a partition of length 2, and the second component in those partitions are k and j, respectively. Establishing this fact for arbitrary n would not only prove that this family of graphs can be distinguished based on their CSF, but it interesting as one would not expect the CSF to capture the difference in the graphs in such a simple and obvious way. We prove this fact next. First we introduce the *vertex distance* $\gamma_G(\cdot,\cdot)$ between two vertices of a graph G, which is simply the number of vertices in the shortest path between them. We define $\gamma_G(u,v) = \infty$ if there is no path between u and v. Where G is clear from context, we omit it in the subscript. For example, if u_1 and u_2 are the two leaves in P_n where $n \geq 2$, then $\gamma_{P_n}(u_1,u_2) = n$.

Lemma 3.1.1. *Fix $n \in \mathbb{Z}^+$ and a tuple $(k,j) \in \{2,3,\dots,\lfloor n/2 \rfloor\} \times \{2,3,\dots,\lceil n/2 \rceil\}$. The unsigned coefficient of $p_{(n-k+1,k)}$ in $X_{P_{n,j}}$, denoted by $[p_{(n-k+1,k)}]_{P_{n,j}}$, is the amount of the four inequalities below that are satisfied.*

$$j \leq k-1 \tag{3.1}$$

$$j \leq n-k \tag{3.2}$$

$$j \geq n-k+2 \tag{3.3}$$

$$j \geq k+1 \tag{3.4}$$

Proof. We wish to apply **Lemma 1.3.3** by finding the number subsets $S \subset E(P_{n,j})$ such that $\lambda(S) = (n-k+1,k)$. Let $\ell \in V(P_{n,j})$ be the leaf that is adjoined to the path graph P_n to construct $P_{n,j}$ (see Figure 3.1). Since $k > 1$, ℓ cannot be in a connected component of vertex size 1 in $P_{n,j}(S)$. Furthermore, if ℓ is in a connected component in $P_{n,j}$ that does not contain any other leaves of $P_{n,j}$, then $P_{n,j}(S)$ must have at least 3 connected components, therefore $l(\lambda(S)) > 2$. Therefore it suffices to find the number of unique subsets $S \subset E(P_{n,j})$ such that in $P_{n,j}(S)$, ℓ is in a connected component of either size k or $n-k+1$ and contains at least one other leaf of $P_{n,j}$. Then, by putting the vertices not in this connected component into another connected component, we have that $\lambda(S) = (n-k+1,k)$.

Let u_1 be the closest leaf to ℓ with respect to γ (if $j = \lceil n/2 \rceil$ and n is odd, we can pick either of the leaves in $P_{n,j}$ that are not ℓ to be u_1). We notice that we can put ℓ in a connected component of size k which contains the leaf u_1 if and only if $\gamma(\ell,u_1) \leq k$. Likewise, we can put ℓ in a connected component of size $n-k+1$ which contains u_1 if and only if $\gamma(\ell,u_1) \leq n-k+1$. By the same logic, we wish to study the inequalities $\gamma(\ell,u_2) \leq k$ and $\gamma(\ell u_2) \leq n-k+1$, where u_2 is the other leaf in $P_{n,j}$. We notice that $\gamma(\ell,u_1) = j+1$ and $\gamma(\ell,u_2) = n-j+2$, giving the inequalities in the statement of this lemma.

Moreover, since $k \leq \lfloor n/2 \rfloor$, $2k < n+1$, therefore $n-k+1 \neq k$. Hence, putting ℓ into a connected component of size (k OR $n-k+1$) containing leaf (u_1 OR u_2) are indeed four distinct cases. □

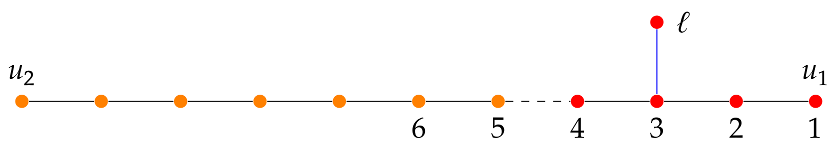


Figure 3.1: An example of a subset $S \subset E(P_{11,3})$ such that $\lambda(S) = (7,5)$. In specific, we can find such a subset since $\gamma(\ell, u_1) = 4 \leq k = 5$.

Theorem 3.1.1. *Fix $n \in \mathbb{Z}^+$ and let $1 \leq k, j < \lceil n/2 \rceil$ and $k \neq j$. Then,*

$$X_{P_{n,k}} - X_{P_{n,j}} = (-1)^{n-1}(p_{(n-k+1,k)} - p_{(n-j+1,j)}) + \sum_{\substack{\lambda \vdash n \\ l(\lambda)>2}} a_\lambda p_\lambda$$

Proof. We prove the result by looking at three disjoint cases for the values of k in the tuples (k, j). In the first case, let $k = 1$ and consider the tuple $(1,1)$, Then, $[p_{(n,1)}]_{P_{n,k}}$ is the number of leaves in $P_{n,k}$. Hence, $[p_{(n,1)}]_{P_{n,k}} = 2$. Furthermore, consider the tuple $(1, j)$, where $1 < j \leq \lceil n/2 \rceil$, then $P_{n,j}$ has 3 leaves, therefore $[p_{(n,1)}]_{P_{n,j}} = 3$. Therefore, the proof is complete with respect to the coefficient of $p_{(n,1)}$, that is,

$$[p_{(n,1)}]_{P_{n,k}} - [p_{(n,1)}]_{P_{n,j}} = \begin{cases} -1 & \text{if } k = 1 \text{ and } 1 < j \leq \lceil n/2 \rceil \\ 0 & \text{otherwise} \end{cases}$$

Suppose now that $1 < k \leq \lfloor n/2 \rfloor$. Then the tuple (k, k) only satisfies inequality (3.2) from **Lemma 3.1.1**. It follows that $[p_{(n-k+1,k)}]_{P_{n,k}} = 1$. Next, suppose that $j = 1$, then $P_{n,j} = P_{n+1}$, and since $k \leq \lfloor n/2 \rfloor$, there are two unique edges in P_{n+1} such that their removal gives a subset $S \subset E(P_{n+1})$ with $\lambda_{P_{n+1}}(S) = (n-k+1, k)$, hence $[p_{(n-k+1,k)}]_{P_{n,j}} = 2$. Thirdly, suppose $1 < j < k$. Then the pair (k, j) satisfies only inequalities (3.1) and (3.2) from **Lemma 3.1.1**. It follows that $[p_{(n-k+1,k)}]_{P_{n,j}} = 2$. Finally, suppose that $k < j \leq \lceil n/2 \rceil$, then the pair (k, j) satisfies only (3.2) and (3.4), therefore we again have $[p_{(n-k+1,k)}]_{P_{n,j}} = 2$. The proof is complete this case.

In the last case, we have $k > \lfloor n/2 \rfloor$, in which case n is odd and $k = (n+1)/2$. In this case, there is no way to put ℓ into a connected component with either of the other leaves in $P_{n,k}$ such that the connected component has size k. Therefore $[p_{(n-k+1,k)}]_{P_{n,k}} = 0$. Moreover, if $j = 1$, then only removing the unique center edge (that is, the edge connecting the two unique centroids of $P_{n,j} = P_{n+1}$) will give a subset $S \subset E(P_{n,j})$ such that $\lambda(S) = (n-k+1, k)$, so $[p_{(n-k+1,k)}]_{P_{n,j}} = 1$. Lastly, if $1 < j \leq \lfloor n/2 \rfloor$, then the tuple (k, j) satisfies only inequalities (3.1) and (3.2). However, since $n-k+1 = k$ in this case, putting γ in a connected component with u_1 of size k or $n-k+1$ is the same connected component, so there is only one unique subset $S \subset E(P_{n,j})$ such that $\lambda(S) = (n-k+1, k)$. Therefore $[p_{(n-k+1,k)}]_{P_{n,j}} = 1$. Finally, we have verified the statement of the theorem in this case. The term $(-1)^{n-1}$ follows from $\#S = n-1$ and **Lemma 1.3.3**. □

As we will see later, studying the algebraic difference in the CSF of two graphs, as we have done here, is a useful technique to understand the differences in the graph structure. In particular, this is an idea that we have not seen in the literature as an approach to Stanley's isomorphism conjecture. More on that idea is given in **Section 3.3**. In fact, our result here is a specific instance of the main result in **Section 3.2**, as the trees we considered here are in-fact spiders. However, our analysis is more careful in giving an explicit characterization of the algebraic difference of the CSF of two trees in this section, which is not done in the next section.

3.2 A Simple Proof that the CSF Distinguishes Spiders

Spiders are a certain infinite family of trees which have at most one vertex with degree greater than 2, call it u. It's not difficult to see that a spider, $S = (V, E)$ is uniquely defined by the length of it's *legs*, that is, the sequence $(\gamma(l, u))_{l \in L(S)}$. In particular, we define a leg to be the path from a leaf in S to the vertex u. In this section, we give a simple, combinatorial proof that spiders can be reconstructed from the chromatic symmetric function.

This result was first proven by Martin, Morin, and Wager [12] by showing that the chromatic symmetric function is a stronger graph invariant than the subtree polynomial (see **Subsection 2.2.2**), and using the properties preserved by the subtree polynomial, namely the degree and path sequence, to show that spiders are distinguished. Later, Crew [3] showed the same result in a different way by using the additional information of the trunk size and twig lengths (see **Subsection 2.2.3**). However, Crew's still proof still relied on the subtree polynomial, and therefore does not provide insight into *how* the CSF can distinguish spiders. By contrast, our proof directly uses the properties of the CSF in the power sum basis to give the result, therefore showing how to reconstruct a spider from X_S. Moreover, it is significantly simpler than the proofs in [12] and [3].

Theorem 3.2.1. *Let $S = (V, E)$ be a spider, then S can be reconstructed from X_S.*

Proof. First, we note that for a subset $S \subset E$, $G(S)$ is a subtree of S containing k vertices adjoined with $n-k$ independent vertices if and only if S is constructed by taking the entire edge set E and trimming a *leaf edge*, that is, an edge such that one of its endpoints is a leaf, iteratively $n-k$ times. By this, we mean removing a leaf edge, then removing a leaf edge in the remaining graph, and so on, $n-k$ times. Moreover, by **Lemma 1.3.3** the number of such subsets S is exactly $[p_{(k,1^{n-k})}]_S$. For example, $[p_{(n-1,1)}]_S$ is the number of leaves of S, which is also the number of ways to remove a leaf edge one time from S. In particular, define $L_1 := [p_{(n-1,1)}]_S$ as the number of legs in S with length at least 1.

Now, consider $[p_{(n-2,1^2)}]_S$. Either we can remove a leaf edge from two different legs or, if a leg has length at least 2, we can remove a leaf edge twice from the same leg. In this way, we have

$$[p_{(n-2,1^2)}] = \binom{L_1}{2} + L_2$$

where L_2 is the number of legs with length at least 2. In the same sense, we can write $L_k = f(L_{k-1}, L_{k-2}, ..., L_1)$ for some explicit expression f, which implores an approach by induction.

Namely, suppose we are given the numbers $(L_1, L_2, ..., L_k)$ and that L_j correctly counts the number of legs in S with length at least j, for $1 \leq j \leq k$. In particular, we can assume that there is at least one leg in S with length greater than k, as else we can already reconstruct S using the given information.

Consider the coefficient $[p_{(n-k-1,1^{k+1})}]_S$, which is the number of ways to iteratively trim $k+1$ leaf edges from S, irrespective of the order of trimming. We note that we trim leaves from at least two different legs if and only if we do not trim more than k leaf edges from any single leg. Moreover, given $L_k, L_{k-1}, ..., L_1$, the number of legs with length exactly j for $1 \leq j < k$ is given by $\alpha_j = L_{j+1} - L_j$, and we define $\alpha_k = L_k$. The number of ways to remove $k+1$ leaf edges from at least two legs is the number of integer solutions to $x_1 + x_2 + ... + x_{L_1} = k+1$ such that the value of x_i does not exceed the length of the i^{th} leg (where we assign an order to the legs arbitrarily). That is, we trim x_i leaf edges from the i^{th} leg. We have,

$$f(\alpha_k, \alpha_{k-1}, ..., \alpha_1, L_1) = \frac{1}{(k+1)!} \frac{d^{k+1}}{dx^{k+1}} \frac{\prod_{i=1}^{k}(1 - x^{\alpha_i+1})}{(1-x)^{L_1}} \Bigg|_{x=0}$$

That is, we isolate the coefficient of x^{k+1} in the generating function for this integer composition problem.[2] Since the cases where we remove $k+1$ leaf edges from a single leg and the cases counted by f are disjoint, we have,

$$L_{k+1} = [p_{(n-k-1,1^{k+1})}]_S - f(L_k, L_{k-1}, ..., L_1)$$

Lastly, we can find the sequence of leg-lengths from $(L_1, L_2, ..., L_\rho)$, where ρ is the length of the longest leg in S, as the number of legs with length exactly j is $L_j - L_{j+1}$, for any $1 \leq j \leq \rho - 1$, which uniquely determines S up to isomorphism. □

Example 3.2.1. Suppose we are given the following CSF of a spider,

$$X_S = -p_{(6)} + 3p_{(5,1)} + 2p_{(4,2)} - 5p_{(4,1^2)} - 4p_{(3,2,1)} + 5p_{(3,1^3)} - p_{(2^3)} + 5p_{(2^2,1^2)} - 5p_{(2,1^4)} + p_{(1^6)}$$

From $[p_{(5,1)}]_S = 3$, we know that S has 3 leaves. From $[p_{(4,1^2)}]_S = 5$, we know that there are $5 - \binom{3}{2} = 2$ legs of length at least 2, hence there is one leg of length 1. Moreover, there are 5 integer solutions to the equation $x_1 + x_2 + x_3 = 3$ such that $0 \leq x_1 \leq 1$ and $0 \leq x_2, x_3 \leq 2$, hence there are $[p_{(3,1^3)}]_S - 5 = 0$ legs of length 3 in S. □

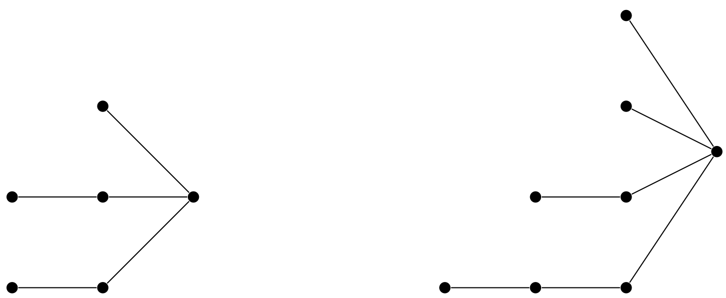


Figure 3.2: Spiders S from **Example 3.1** (left) and **Example 3.2** (right).

Example 3.2.2. Suppose we are given the following CSF of a spider,

$$X_S = 4p_{(7,1)} + 8p_{(6,1^2)} + 11p_{(5,1^3)} + 10p_{(4,1^4)} + 9p_{(3,1^5)} + 7p_{(2,1^6)} + \sum_{\substack{\lambda \vdash n \\ \lambda \neq (k,1^{8-k})}} a_\lambda p_\lambda$$

From $[p_{(7,1)}] = 4$, we know that S has 4 leaves. From $[p_{(6,1^2)}] = 8$, we know that there are $8 - \binom{4}{2} = 2$ legs of length at least 2, hence there are two legs of length 1. There are 10 integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 3$ such that $0 \leq x_1, x_2 \leq 1$ and $0 \leq x_3, x_4 \leq 2$. Hence, there are $[p_{(5,1^3)}]_S - 10 = 1$ legs of length at least 3. Therefore there is one leg of length 3. □

[2]There are many ways to do this counting, but we use an approach by generating functions since it lends itself to efficient computation and, for our purposes, there is no need for a closed-form solution.

3.3 A Forest-Classification of the Difference of Two CSFs

As we have seen in **Section 3.1**, given the chromatic symmetric function of two trees, X_{T_1} and X_{T_2} it is useful to analyze the symmetric function $X_{T_1} - X_{T_2}$ in order to understand the relative differences in the structure of the two trees. In this section, we give a characterization of the function $X_{T_1} - X_{T_2}$ in-terms of a sum of chromatic symmetric functions of forest graphs. In fact, there is no intuitive reason to expect that the difference in the CSF of two trees corresponds to the CSF of a graph at all. Furthermore, we end with a discussion about Stanley's isomorphism conjecture, and how the ideas used leading to our main result can be applied to prove the conjecture.

Before we can prove our main result, we first need the following lemma of Orellana and Scott.

Lemma 3.3.1 ([13]). *Let $G_{1,2}$ be a graph with the adjacent edges $e_1 = (v, v_1)$, $e_2 = (v, v_2)$ and $e_3 = (v_1, v_2) \notin E(G_{1,2})$. Define,*

$$
\begin{aligned}
G_{1,3} &= (V(G_{1,2}), (E(G_{1,2}) - \{e_2\}) \cup \{e_3\}) \\
G_{2,3} &= (V(G_{1,2}), (E(G_{1,2}) - \{e_1\}) \cup \{e_3\}) \\
G_1 &= (V(G_{1,2}), E(G_{1,2}) - \{e_2\}) \\
G_3 &= (V(G_{1,2}), (E(G_{1,2}) - \{e_1, e_2\}) \cup \{e_3\})
\end{aligned}
$$

Then, $X_{G_{1,2}} = X_{G_{2,3}} + X_{G_1} - X_{G_3}$.

Proof. See **Corollary 3.2** in [13], pg. 6. □

This lemma states that by shifting an edge in a graph, that is, changing the graph $G_{1,2}$ to the graph $G_{2,3}$, the CSF correspondingly shifts by a factor of $X_{G_1} - X_{G_3}$. Our main idea is to repeatedly apply this "shifting" operation to completely transform one tree into another. However, some preliminary work needs to be done to justify that such a transformation is even possible. The main technical barrier to overcome is handled in the following lemma.

Lemma 3.3.2. *Let $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$ be two labeled trees such that the correspondence $u_k \leftrightarrow v_k$ is not an isomorphism. Then, there are always numbers $1 \le i_1 < j_1 \le n$ and $1 \le i_2 < j_2 \le n$ such that,*

1. *$(u_{i_1}, u_{j_1}) \in E_1$ but $(v_{i_1}, v_{j_1}) \notin E_2$, and $(u_{i_2}, u_{j_2}) \notin E_1$ but $(v_{i_2}, v_{j_2}) \in E_2$*
2. *Either[3] the unique path from u_{i_1} to u_{i_2} in T_1 does not contain u_{j_1}, or the unique path from u_{i_1} to u_{j_2} in T_1 does not contain u_{j_1}.*

Proof. We first prove the existence numbers (i_1, i_2, j_1, j_2) that satisfy (1.). Then, we show that if this choice does not satisfy (2.), then this implies must be another choice that does satisfy (1.) and (2.).

Since the correspondence $u_k \leftrightarrow v_k$ is not an isomorphism, there must exist $1 \le i_1 < j_1 \le n$ such that the edge $(u_{i_1}, u_{j_1}) \in E_1$ but $(v_{i_1}, v_{j_1}) \notin E_2$. Moreover, this implies that there are numbers $1 \le i_2 < j_2 \le n$ such that $(u_{i_2}, u_{j_2}) \notin E_1$ but $(v_{i_2}, v_{j_2}) \in E_2$. This fact follows directly from the property that $\#E_1 = \#E_2 = n - 1$.

Now, suppose the choice of (i_1, i_2, j_1, j_2) as above does not satisfy (2.). That is, with generality up to switching i_1 and j_1, both the unique path from u_{i_1} to u_{i_2} in T_1 and the unique path from u_{j_1}

[3]Exclusive, i.e. *XOR* or ⊕.

to u_{j_2} in T_1 contain the vertex u_{j_1}. Then, by adding the edge (u_{i_2}, u_{j_2}) to E_1, we create the cycle $u_{j_1} \to u_{i_2} \to u_{j_2} \to u_{j_1}$. Therefore, in order to preserve the acyclic property in T_2, there must be at least one pair of numbers $1 \leq i_1' < j_1' \leq n$ such that the edge $(u_{i_1'}, u_{j_2'}) \in E_1$ is either on the unique path from u_{j_1} to u_{i_2} or the path from u_{j_1} to u_{j_2} in T_1, and $(v_{i_1'}, v_{i_2'}) \notin E_2$. Finally, since the edge $(u_{i_1'}, u_{j_1'})$ is on the unique path in T_1 between u_{i_2} and u_{j_2}, the choice of numbers (i_1', i_2, j_1', j_2) must satisfy both properties (1.) and (2.). □

Before continuing in the proof of our main result, we define the function $\text{Agr} : \mathscr{T}_n \times \mathscr{T}_n \to \mathbb{Z}^+$ (Agr for "agreement") where $\mathscr{T}$ is the set of labeled trees on n vertices,[4] such that for $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$, $\text{Agr}(T_1, T_2)$ is the number of ordered tuples (i, j) with $i < j$ such that $(u_i, u_j) \in E_1$ but $(v_i, v_j) \notin E_2$. In particular, we have that if $\text{Agr}(T_1, T_2) = 0$ then T_1 is isomorphic to T_2, however the converse may not be true.

The next lemma is the primary ingredient of our main result, and defines one "step" in the algorithm which transforms T_1 into T_2.

Lemma 3.3.3. *Let $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$ be two labeled trees such that $Agr(T_1, T_2) = m > 0$. Then, there is an algorithm which produces a tree $T = (V = (w_1, w_2, ..., w_n), E)$ such that $Agr(T_1, T) = 1$ and $Agr(T, T_2) = m - 1$. Moreover,*

$$X_{T_1} = X_T + \sum_{i=1}^{\ell_1+\ell_2} X_{F_i^1} - X_{F_i^2}$$

where F_i^1, F_i^2 are forests, and ℓ_1, ℓ_2 are path lengths which are explained below.

Proof. We have that $\text{Agr}(T_1, T_2) = 0$ if and only if the correspondence $u_k \leftrightarrow v_k$ is an isomorphism between T_1 and T_2. Therefore, from **Lemma 3.3.2**, let (i_1, i_2, j_1, j_2) be numbers which satisfy properties (1.) and (2.). We will construct an algorithm which replaces the edge $(u_{i_1}, u_{j_1}) \in E_1$ with the edge (u_{i_2}, u_{j_2}), keeping the other edges in E_1 the same, and call this new graph T.

Without loss of generality, suppose the unique path from u_{i_1} to u_{i_2} does not contain u_{j_1}, and call this path $h_1 = (u_{i_1} = \alpha_0, \alpha_1, ..., \alpha_{\ell_1} = u_{i_2})$. Then, from (2.) in **Lemma 3.3.2**, we also have that the unique path from u_{i_1} to u_{j_2} does contain u_{j_1}, therefore the unique path from u_{j_1} to u_{j_2} does not contain u_{i_1}, and we call this path $h_2 = (u_{j_1} = \beta_0, \beta_1, ..., \beta_{\ell_2} = u_{j_2})$.

The first step in the algorithm is to apply **Lemma 3.3.1** to the edges $e_1 = (\alpha_0, u_{j_1})$ and $e_2 = (\alpha_0, \alpha_1)$. The result is (1) a tree $T^1 = (V_1, E_1^1)$ where E_1^1 is the same as E_1, except the edge (α_0, u_{j_1}) is replaced by the edge (α_1, u_{j_1}), (2) a forest $F^{(1,1)}$ such that $E(F^{(1,1)})$ is the same as E_1 with the edge e_2 removed, and (3) a forest $F^{(1,2)}$ such that $E(F^{(1,2)})$ is the same as E_1 without the edges e_1 or e_2, but with the edge (α_1, u_{j_1}) adjoined. Moreover, $X_{T_1} = X_T + X_{F^{(1,1)}} - X_{F^{(1,2)}}$. We continue by letting $r = 1$, $e_1 = (\alpha_r, u_{j_1})$ and $e_2 = (\alpha_r, \alpha_{r+1})$, then applying **Lemma 3.3.1** on T^r, and incrementing r. The algorithm terminates when $r = \ell_1 + 1$. Finally, we have constructed graphs $\{T^r\}_{1 \leq r \leq \ell_1}$ and forests $\{F^{(r,1)}\}_{1 \leq r \leq \ell_1}$, $\{F^{(r,2)}\}_{1 \leq r \leq \ell_1}$ such that, $T^{\ell_1} = (V_1, E_1^r)$ where E_1^r is the same as E_1, except the edge (u_{i_1}, u_{j_1}) is replaced by the edge (u_{i_2}, u_{j_1}). Moreover, we have,

$$X_{T_1} = X_{T^{\ell_1}} + \sum_{r=1}^{\ell_1} X_{F^{(r,1)}} - X_{F^{(r,2)}}$$

[4] The popular "Cayley's Formula", named for Arthur Cayley, states that $\#\mathscr{T}_n = n^{n-2}$.

We then apply this entire algorithm again to T^{ℓ_1}, but the role of u_{j_1} is replaced by the vertex u_{i_2}, and the role of h_1 is replaced by h_2. As a result, we get a tree $T = (V, E)$ where E is the same as E_1, except the edge (u_{i_1}, u_{j_1}) is replaced by the edge (u_{i_2}, u_{j_2}), hence, $\mathrm{Agr}(T_1, T) = 1$ and $\mathrm{Agr}(T, T_2) = m - 1$. Moreover,

$$X_{T_1} = \left(X_T + \sum_{r=1}^{\ell_2} X_{G^{(r,1)}} - X_{G^{(r,2)}}\right) + \sum_{i=1}^{\ell_1} X_{F^{(r,1)}} - X_{F^{(r,2)}}$$

where $G^{(r,j)}$, $1 \le r \le \ell_2$, $j \in \{1, 2\}$, is a forest on $n-2$ edges. □

It is now simple to prove our main result.

Theorem 3.3.1. *Let $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$ be two labeled trees. Then there is an algorithm which produces two sets of forests $\mathscr{F}_1, \mathscr{F}_2$, each with precisely $n-2$ edges, such that,*

$$X_{T_1} = X_{T_2} + \sum_{\substack{F_1 \in \mathscr{F}_1 \\ F_2 \in \mathscr{F}_2}} X_{F_1} - X_{F_2}$$

Proof. If $\mathrm{Agr}(T_1, T_2) = 0$, then T_1 and T_2 are isomorphic, therefore $X_{T_1} = X_{T_2}$. Otherwise, $\mathrm{Agr}(T_1, T_2) = m > 0$, and we apply the algorithm from **Lemma 3.3.3** iteratively m times to get the result. □

This result is well-motivated in the search for a proof of Stanley's isomorphism conjecture. Namely, one can consider an approach by induction on the number of vertices in a tree. We can assume that Stanley's conjecture is true for trees with fewer than n vertices, and that we are given that $X_{T_1} = X_{T_2}$, where T_1 and T_2 are trees on n vertices. Then, after labeling the trees arbitrarily and running the algorithm from **Lemma 3.3.3**, we must have that,

$$\sum_{F_1 \in \mathscr{F}_1} X_{F_1} = \sum_{F_2 \in \mathscr{F}_2} X_{F_2}$$

Moreover, we have the following result of Wang, Yu, and Zhang [17].

Lemma 3.3.4 ([17]). *Let $m, r \in \mathbb{Z}^+$. Let T be a forest with components $T_1, ..., T_m$ and F be a forest with components $F_1, ..., F_r$. Suppose $X_T = X_F$. Then $m = r$, and there exists a permutation τ of $[m]$ such that $X_{T_i} = X_{F_{\tau(i)}}$ for $i \in [m]$.*

Proof. See **Lemma 2.2** in [17], pg. 4. □

Therefore, if we can give a one-to-one correspondence between the CSFs of forests in $\mathscr{F}_1$ and $\mathscr{F}_2$, we can say that the forests themselves are isomorphic, based on our inductive assumption. From there, it *should* be simple to tell that the two original trees are isomorphic. As evidence for this, a classical result states that trees are reconstructible from only the *deck* of subgraphs admitted by removing a leaf [8], and in our case we only wish to tell two given trees are the same; the reconstruction is unnecessary. As a "proof of concept", we give the following simple result where the correspondence mentioned above is forced, since $\#\mathscr{F}_1 = \#\mathscr{F}_2 = 1$.

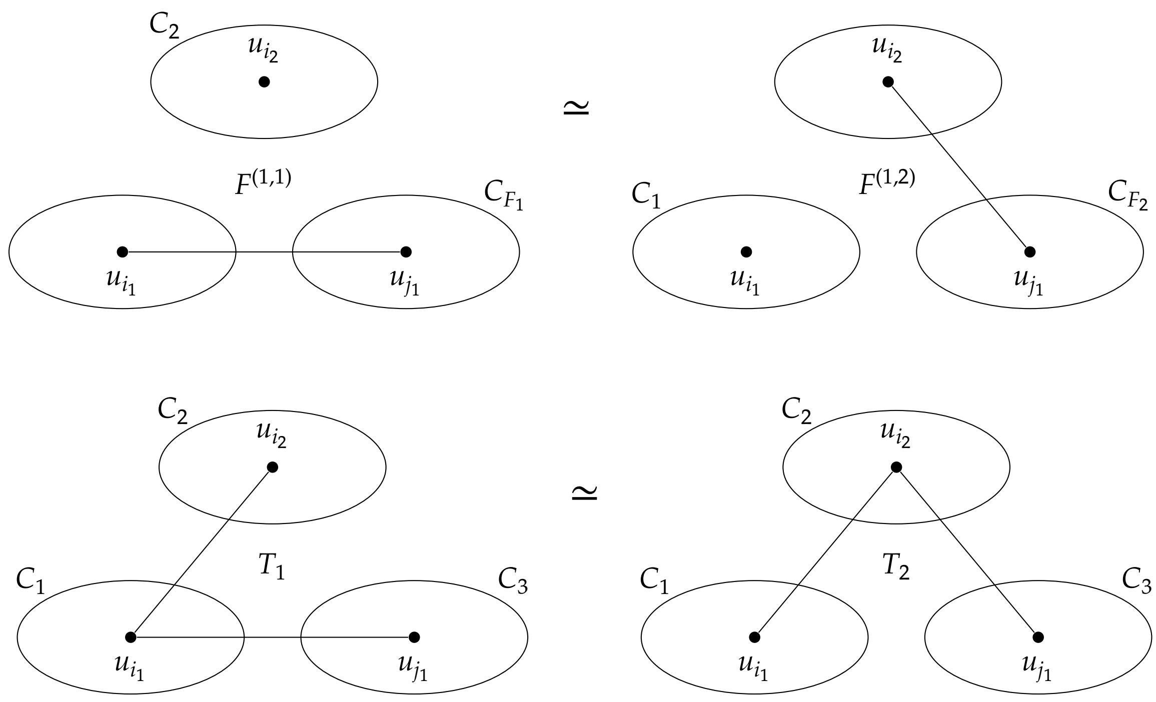


Figure 3.3: Representation of $F^{(1,1)}$ and $F^{(1,2)}$ from **Lemma 3.3.5**. In particular, it is not hard to see that $C_{F_1} \simeq C_{F_2}$ and $C_1 \simeq C_2$ implies that $T_1 \simeq T_2$.

Lemma 3.3.5. *Suppose $X_{T_1} = X_{T_2}$ where $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$. Moreover, suppose that $Arg(T_1, T_2) = 1$, $\ell_1 = 1$, $\ell_2 = 0$, where ℓ_1 and ℓ_2 are defined as in the proof of* **Lemma 3.3.3**, *and Stanley's conjecture is true for trees with $\leq n$ vertices. Then T_1 is isomorphic to T_2.*

Proof. There is only one choice of (i_1, j_1, i_2, j_2) that satisfies both (1.) and (2.) in **Lemma 3.3.2.** Moreover, because $\ell_2 = 0$, we have that $j_1 = j_2$. Consider the graph $T = (V, E)$ where $E = E_1 - \{(u_{i_1}, u_{j_1}), (u_{i_1}, u_{i_2})\}$. Then T has exactly three connected components. Let C_1 be the connected component of T containing u_{i_1}, C_2 be the connected component of T containing u_{i_2}, and C_3 be the connected component of T containing u_{j_1}.

After running the algorithm in **Lemma 3.3.3**, we have that $X_{F^{(1,1)}} = X_{F^{(1,2)}}$, where $F^{(1,1)}$ has one connected component C_1 and C_3 adjoined by the edge (u_{i_1}, u_{i_2}), call it C_{F_1}, and the other connected component C_2. Likewise, $F^{(1,2)}$ has one connected component C_2 and C_3 adjoined by the edge (u_{i_1}, u_{j_1}), call it C_{F_2}, and the other connected component C_1.

By **Lemma 3.3.4** and by matching based on the number of vertices (i.e. C_2 cannot be isomorphic to C_{F_2} since C_{F_2} strictly contains C_2, and likewise $C_1 \not\simeq C_{F_1}$), we have that $X_{C_{F_1}} = X_{C_{F_2}}$ and $X_{C_1} = X_{C_2}$. Moreover, since we are assuming Stanley's conjecture is true for all trees with fewer than n vertices and C_{F_i}, C_j are trees with fewer than n vertices for $i, j \in \{1, 2\}$, $C_{F_1} \simeq C_{F_2}$ and $C_1 \simeq C_2$. From here, it is not hard to see that $T_1 \simeq T_2$ (see Figure 3.3 for a visualization). □

We can consider **Lemma 3.3.5** to be the base case of the following result, which is itself the base case of an inductive proof of Stanley's conjecture.

Lemma 3.3.6. *Suppose $X_{T_1} = X_{T_2}$ where $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$. Moreover, suppose that $Arg(T_1, T_2) = 1$ and Stanley's conjecture is true for trees with $\leq n$ vertices. Then T_1 is isomorphic to T_2.*

Proof outline. We apply induction to the value of ℓ_1. The base case is covered in **Lemma 3.3.5**. Once we have established the statement of **Lemma 3.3.5** for an arbitrary $\ell_1 = m$, a symmetric argument establishes the result for an arbitrary ℓ_2, since the algorithm invoked is the same (see **Lemma 3.3.3**). □

Now, **Lemma 3.3.6** acts as the base case in a proof of Stanley's conjecture based on induction of the value of $\text{Agr}(T_1, T_2)$.

Theorem 3.3.2 (Stanley's isomorphism conjecture). *If $X_{T_1} = X_{T_2}$, then T_1 is isomorphic to T_2.*

Proof outline. We apply induction to the number of vertices in T_1 and T_2. Namely, suppose that for all trees with fewer than n vertices, the statement holds true. For the base case, the conjecture has already been verified computationally for trees up to 29 vertices [9].

We give an arbitrary labeling to the trees T_1 and T_2 such that $T_1 = (V_1 = (u_1, u_2, ..., u_n), E_1)$ and $T_2 = (V_2 = (v_1, v_2, ..., v_n), E_2)$. We apply induction to the value of $\text{Agr}(T_1, T_2)$. The base case of $\text{Agr}(T_1, T_2) = 1$ is handled in **Lemma 3.3.6**. Moreover, suppose that for $\text{Arg}(T_1, T_2) < m$, if $X_{T_1} = X_{T_2}$, then T_1 is isomorphic to T_2. For the inductive step, we wish to show that the result holds for $\text{Agr}(T_1, T_2) = m$.

Then, for any arbitrary labeled graphs T_1 and T_2 on n vertices, there is some natural number k less than n such that $\text{Agr}(T_1, T_2) = k$, therefore the result holds in general that $X_{T_1} = X_{T_2}$ implies T_1 is isomorphic to T_2. Stanley's conjecture follows by induction on n. □

We leave it as future work to ourselves, and more broadly to the many researchers working on Stanley's conjecture, to complete the two inductive steps left unfinished in the proof outlines of **Lemma 3.3.6** and **Theorem 3.3.2**.

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I am honored to have been advised by Dr. Italo Simonelli, and in particular to have the privilege of being in the final cohort of his advisees with respect to such *Signature Work* theses. I thank him for his dedicated mentorship throughout the course of this project, and will forever cherish his insight in mathematics, life, and the ruthless underside of early 20th century Hungarian mathematics.

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SIGNATURE WORK NARRATIVE

The study of chromatic symmetric functions lies very clearly in the field of algebraic combinatorics, which uses methods from combinatorics to study topics in algebra such as group theory, algebraic geometry, and algebraic topology. As such, it is not surprising that two of the three thematic courses I have taken which provided a strong foundation for my completion of this work are (1) **Discrete Math for Computer Science** (COMPSCI203), which covers the basic of combinatorics and graph theory, and touches on some topics in algebraic combinatorics such as generating functions, and (2) **Abstract Algebra** (MATH401), which introduces the mathematical study of algebra, including group theory and applications to counting. In COMPSCI203, we study how to write combinatorial proofs in this course, which appear many times throughout this project, as well as fundamental combinatorial objects like binomial coefficients, multinomial coefficients, Stirling numbers of the 1^{st} and 2^{nd} kind, Catalan numbers, and more. Alternatively, the set of symmetric functions homogeneous of degree n form a commutative ring, which is an algebraic structure studied extensively in MATH401. Understanding rings and their properties are fundamental to understanding operations on symmetric functions, such as the so-called ”Hall inner-product” of symmetric functions.

My third thematic course is (3) **Probability and Statistics** (MATH205), the elements discussed in which appear, for example, in **Chapter 1** in the proof of the so-called ”hook length formula”. In fact, this method of giving a probabilistic interpretation of some key numbers and using normalization to prove they must add to 1 was used to prove the other, major open problem in the study of the CSF, called the ”Stanley-Stembridge Conjecture” [10]. Understanding these proofs requires an understanding of the fundamentals of probability, at least at the level of applications.

Given my interest in the topics covered by these three courses, I have found opportunities to engage in the course material after my enrollment in the courses terminated. Throughout these experiences, I was able to further solidify my interest in these topics, leading to my pursuit of the chromatic symmetric function as a topic of study, as well as discuss relevant ideas and brainstorm alongside faculty and peers at DKU. In the case of COMPSCI203, in the semester following my enrollment in this course, I was invited by Professor Xing Shi Cai to study alongside 3 other DKU peers a textbook entitled ”Concrete Mathematics” by Donald Knuth. This textbook directly extended the course materials of COMPSCI203, and we met weekly to discuss one chapter per meeting, with a different student presenting each week. In particular, I recall presenting on the chapter regarding generating functions, which are a formal power series, just like symmetric functions, and which deepened my interest in the field of algebraic combinatorics. Beyond this reading group, I have had the privilege of serving as the teaching assistant (TA) for this course under Professor Cai for two semesters (Spring 2024 and Spring 2025).

Regarding Abstract Algebra (MATH401), under the guidance of Professor Italo Simonelli, several students and I began a venture in Fall 2024 to read-through the textbook ”Algebra” by Emil

Artin, which is a more advanced extension of the course materials from MATH401. In particular, we adopted the same meeting style as the group led by Professor Cai, and are still meeting regularly, discussing one chapter in each meeting. In specific, one very interesting topic that I have discovered through this reading group is the application of group actions to counting, that is, using algebra to study combinatorics.

Moreover, as the TA for MATH206[5] under Professor Pascal Grange in the current semester (Spring 2025), I have found an opportunity to engage with the course materials by assisting other students in comprehending the abstract ideas covered. In particular, I have found that my own understanding of the course materials has significantly deepened as to explain something in clear terms requires a strong understanding. Moreover, revisiting these materials after taking the course Measure and Integration (MATH450) gives me a new perspective. Namely, a function which assigns a probability to each event in an event space is a measure assigned to that set. Therefore the results from general measure theory can be applied to probability theory. For example, I now understand the derivations of results such as Chebyshev's inequality and the Borel-Cantelli lemma, which are stated, but not proven, in MATH206.

Finally, one of the most important ways that I have engaged with the DKU math community is through the "Discrete Math Seminar", which has been held semi-regularly for many years, now. In Spring 2023, I was invited by Professor Simonelli to give a presentation at the seminar, my chosen topic for-which was "Benford's Law". I found it very exciting to dive deeply into this topic and design a comprehensive review of the results in the study of Benford's Law that I found most interesting. Furthermore, I was able to apply my knowledge in other courses, such as my final project for Numerical Analysis (MATH302). This experience inspired me to deepen my engagement with the seminar, not only helping to organize the seminar in the following semesters, but also giving talks whenever the opportunity arose, covering the method of finite differences in Fall 2023, and a research project I completed at another institution about private algorithms to release graph properties in Fall 2024. I accredit the Discrete Math Seminar and the dedicated faculty in DKU's math department with my continued interest in pursuing research in mathematics and theoretical computer science, and moreover my interest in this project.

[5]formerly MATH205