Spiders are Reconstructible from their Chromatic Symmetric Functions

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Abstract

Spiders are a certain infinite family of trees which have at most one vertex with degree greater than 2. It was first shown by Martin, Morin, and Wager [4] that the chromatic symmetric function (CSF) distinguishes spiders by showing that the CSF is a stronger graph invariant than the subtree polynomial, and using the properties preserved by the subtree polynomial, namely the degree and path sequence. Later, Crew [1] showed the same result in a different way by using the additional information of the trunk size and twig lengths. However, Crew's proof still relied on the subtree polynomial, and therefore does not provide insight into *how* the CSF can distinguish spiders. In this note, we give a proof which directly uses the properties of the CSF in the power sum basis to give the result, therefore showing how to reconstruct a spider from its CSF.

1 Introduction

The chromatic symmetric function (CSF) was introduced by Richard Stanley in his 1995 seminal work, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph* [5]. Given a graph $G = (V, E)$, the CSF of G is denoted as X_G and is defined as a sum over all proper colorings $\phi : V \to \mathbb{N}$ of G, and is defined by,

$$X_G = \sum_{\phi} \mathbf{x}^{\phi} = \sum_{\phi} \prod_{v \in V} x_{\phi(v)}$$

Before we move-on, we state but do not prove one main result about the CSF, which gives a combinatorial interpretation to its coefficients when expressed in the power sum basis. As a bit of notation, if $G = (V, E)$ is a graph and $S \subset E$, then we denote by $G(S) = (V, S)$ the subgraph of G with edge set S. Moreover, we denote by $\lambda_G(S)$ the partition of $n = |V|$ formed by sorting the number of vertices in each connected component of $G(S)$ in weakly decreasing order. Where G is obvious, we omit from the subscript and simply write $\lambda(S)$. We are now ready to state the result.

Lemma 1.1 ([5]).

$$X_G = \sum_{S \subset E} (-1)^{|S|} p_{\lambda(S)}$$

Stanley not only introduced the chromatic symmetric function in 1995, but also stated two big questions, motivating its further study. The first asked for sufficient conditions for the CSF to the e-positive, that is, have positive coefficients in the elementary symmetric function basis, called the "Stanley-Stembridge Conjecture", and has recently been solved [3][1]. The other still remains open to this day, and is commonly dubbed *Stanley's isomorphism conjecture*, *Staney's tree conjecture*, or *Stanley's tree isomorphism conjecture*.

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[1] It suffices for G to be the incomparability graph of a "claw free" poset, that is, a $(3 + 1)$-free poset.

In particular, Stanley proposed that two trees are isomorphic if and only if they have the same CSF. So far, this conjecture is widely believed to be true, and has been verified for trees up to 29 vertices [2]. Proving the conjecture would have applications to many far-reaching areas of mathematics and beyond. For example, it would allow us to efficiently implement algorithms to check for tree isomorphisms by checking any of the characterizations of the coefficients for the CSF of trees in any symmetric function basis, such as the number of independent sets corresponding to each partition of n.

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2 Main Result

It's not difficult to see that a spider, $S = (V, E)$ is uniquely defined by the length of it's *legs*, that is, the sequence $(\gamma(l, u))_{l \in L(S)}$, where $\gamma : V \times V \to \mathbb{N}$ is the vertex distance between two vertices (i.e. the number of vertices on the shortest path between them). In particular, we define a leg to be the path from a leaf in S to the vertex u. In this section, we give a simple, combinatorial proof that spiders can be reconstructed from the chromatic symmetric function.

Theorem 2.1. *Let $S = (V, E)$ be a spider, then S can be reconstructed from X_S.*

Proof. First, we note that for a subset $S \subset E$, $G(S)$ is a subtree of S containing k vertices adjoined with $n-k$ independent vertices if and only if S is constructed by taking the entire edge set E and trimming a *leaf edge*, that is, an edge such that one of its endpoints is a leaf, iteratively $n-k$ times. By this, we mean removing a leaf edge, then removing a leaf edge in the remaining graph, and so on, $n-k$ times. Moreover, by **Lemma 1.1** the number of such subsets S is exactly $[p_{(k,1^{n-k})}]_S$. For example, $[p_{(n-1,1)}]_S$ is the number of leaves of S, which is also the number of ways to remove a leaf edge one time from S. In particular, define $L_1 \coloneqq [p_{(n-1,1)}]_S$ as the number of legs in S with length at least 1.

Now, consider $[p_{(n-2,1^2)}]_S$. Either we can remove a leaf edge from two different legs or, if a leg has length at least 2, we can remove a leaf edge twice from the same leg. In this way, we have

$$[p_{(n-2,1^2)}] = \binom{L_1}{2} + L_2$$

where L_2 is the number of legs with length at least 2. In the same sense, we can write $L_k = f(L_{k-1}, L_{k-2}, ..., L_1)$ for some explicit expression f, which implores an approach by induction.

Namely, suppose we are given the numbers $(L_1, L_2, ..., L_k)$ and that L_j correctly counts the number of legs in S with length at least j, for $1 \le j \le k$. In particular, we can assume that there is at least one leg in S with length greater than k, as else we can already reconstruct S using the given information.

Consider the coefficient $[p_{(n-k-1,1^{k+1})}]_S$, which is the number of ways to iteratively trim $k+1$ leaf edges from S, irrespective of the order of trimming. We note that we trim leave edges from at least two different legs if and only if we do not trim more than k leaf edges from any single leg. Moreover, given $L_k, L_{k-1}, ..., L_1$, the number of legs with length exactly j for $1 \le j < k$ is given by

$\alpha_j = L_{j+1} - L_j$, and we define $\alpha_k = L_k$. The number of ways to remove $k+1$ leaf edges from at least two legs is the number of integer solutions to $x_1 + x_2 + ... + x_{L_1} = k+1$ such that the value of x_i does not exceed the length of the i^{th} leg (where we assign an order to the legs arbitrarily). That is, we trim x_i leaf edges from the i^{th} leg. We have,

$$f(\alpha_k, \alpha_{k-1}, ..., \alpha_1, L_1) = \frac{1}{(k+1)!} \frac{d^{k+1}}{dx^{k+1}} \frac{\prod_{i=1}^{k}(1 - x^{\alpha_i + 1})}{(1-x)^{L_1}} \Bigg|_{x=0}$$

That is, we isolate the coefficient of x^{k+1} in the generating function for this integer composition problem.[2] Since the cases where we remove $k+1$ leaf edges from a single leg and the cases counted by f are disjoint, we have,

$$L_{k+1} = [p_{(n-k-1,1^{k+1})}]_S - f(L_k, L_{k-1}, ..., L_1)$$

Lastly, we can find the sequence of leg-lengths from $(L_1, L_2, ..., L_\rho)$, where ρ is the length of the longest leg in S, as the number of legs with length exactly j is $L_j - L_{j+1}$, for any $1 \leq j \leq \rho - 1$, which uniquely determines S up to isomorphism. □

Example 2.1. Suppose we are given the following CSF of a spider,

$$X_S = -p_{(6)} + 3p_{(5,1)} + 2p_{(4,2)} - 5p_{(4,1^2)} - 4p_{(3,2,1)} + 5p_{(3,1^3)} - p_{(2^3)} + 5p_{(2^2,1^2)} - 5p_{(2,1^4)} + p_{(1^6)}$$

From $[p_{(5,1)}]_S = 3$, we know that S has 3 leaves. From $[p_{(4,1^2)}]_S = 5$, we know that there are $5 - \binom{3}{2} = 2$ legs of length at least 2, hence there is one leg of length 1. Moreover, there are 5 integer solutions to the equation $x_1 + x_2 + x_3 = 3$ such that $0 \leq x_1 \leq 1$ and $0 \leq x_2, x_3 \leq 2$, hence there are $[p_{(3,1^3)}]_S - 5 = 0$ legs of length 3 in S. □

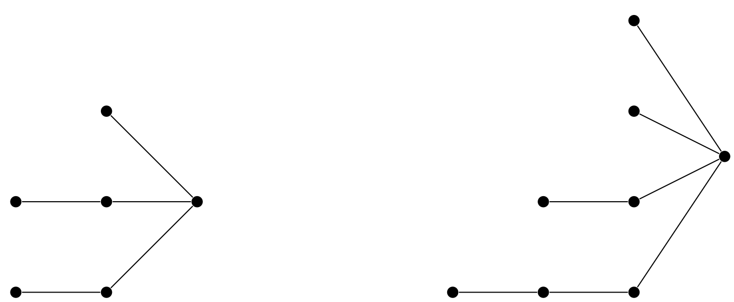

Figure 1: Spiders S from **Example 3.1** (left) and **Example 3.2** (right).

Example 2.2. Suppose we are given the following CSF of a spider,

$$X_S = 4p_{(7,1)} + 8p_{(6,1^2)} + 11p_{(5,1^3)} + 10p_{(4,1^4)} + 9p_{(3,1^5)} + 7p_{(2,1^6)} + \sum_{\substack{\lambda \vdash n \\ \lambda \neq (k,1^{8-k})}} a_\lambda p_\lambda$$

From $[p_{(7,1)}] = 4$, we know that S has 4 leaves. From $[p_{(6,1^2)}] = 8$, we know that there are $8 - \binom{4}{2} = 2$ legs of length at least 2, hence there are two legs of length 1. There are 10 integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 3$ such that $0 \leq x_1, x_2 \leq 1$ and $0 \leq x_3, x_4 \leq 2$. Hence, there are $[p_{(5,1^3)}]_S - 10 = 1$ legs of length at least 3. Therefore there is one leg of length 3. □

[2] There are many ways to do this counting, but we use an approach by generating functions since it lends itself to efficient computation and, for our purposes, there is no need for a closed-form solution.

References

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