Finite Calculus: A Discrete Analouge to Infinite Calculus

Discrete Math Seminar

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Outline

Motivation

A Nasty Sum

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$$\int_0^n x \ln x = \frac{1}{2}n^2\left(\ln n - \frac{1}{2}\right)$$

Real-Life Sums

1. Average codeword length (n bits),

$$\frac{\sum_{k=1}^{n} k2^k}{\sum_{k=1}^{n} 2^k}$$

2. Sum of exponents

$$\sum_{k=1}^{n} k^m \text{ , } m \in \mathbb{Z}$$

3. Number of comparisons in quick sort (n items)

$$2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}$$

Finite Calculus

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We have already seen that

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Let us find some more analogues between finite and infinite calculus.

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Finite calculus,

$$\Delta x^{\underline{m}} = (x+1)x(x-1)...(x-(m-2)) - x(x-1)...(x-(m-1)) = (x+1)x^{\underline{m-1}} - (x-(m-1))x^{\underline{m-1}} = mx^{\underline{m-1}}$$

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Finite calculus,

$$\sum\nolimits_a^b g(x)\ \delta x := G(b) - G(a)$$

Definite Anti-Difference

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$$\sum\nolimits_a^b g(x)\ \delta x = g(b-1) + g(b-2) + ... + g(a) = \sum_{k=a}^{b-1} g(k)$$

Negative Falling Powers

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Which implies,

$$\sum u\Delta v\ \delta x = uv + \sum Av\Delta u\ \delta x$$

Finite Calculus Summary

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$\int uv'\ dx = uv + \int vu'\ dx$	$\sum u\Delta v\ \delta x = uv + \sum Av\Delta u\ \delta x$

Motivating Example

Let us return to our original example,

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It is now not hard to see that

$$\sum_{k=0}^{n} xH_x = \frac{1}{2}n(n-1)\left(H_n - \frac{1}{2}\right)$$

Practice Problem

Your turn! What is,

$$\sum_{k=0}^{n-1} \binom{k}{m} H_k = ???$$

when m is a nonnegative integer?

$$\text{Hint 1: } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\text{Hint 2: } \sum_{k=0}^{n-1} \binom{k+1}{m+1} \frac{1}{k+1} = \sum_{k=0}^{n-1} \binom{k}{m} \frac{1}{m+1} = \binom{n}{m+1} \frac{1}{m+1}$$

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Answer: $\binom{n}{m+1}\left(H_n - \frac{1}{m+1}\right)$

Motivation

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How do we convert between **falling powers** and **exponents**?

Stirling Numbers

Stirling numbers of the second kind,

$$\left\{ {n \atop k} \right\}\text{: "n subset k"}$$

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$$\text{Example: } \left\{ {3 \atop 2} \right\} = 3$$

Stirling Number Recurrence

Combinatorial proof that,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

Connection to Falling Powers

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$$\begin{aligned} x^{n+1} = x x^n = x \sum_k \left\{ n \atop k \right\} x^{\underline{k}} = \sum_k \left\{ n \atop k \right\} x x^{\underline{k}} \\ = \sum_k \left\{ n \atop k \right\} \left(x^{\underline{k+1}} + k x^{\underline{k}} \right) = \sum_k \left\{ n \atop k-1 \right\} x^{\underline{k}} + \sum_k k \left\{ n \atop k \right\} x^{\underline{k}} \\ = \sum_k \left(\left\{ n \atop k-1 \right\} + k \left\{ n \atop k \right\} \right) x^{\underline{k}} = \sum_k \left\{ n+1 \atop k \right\} x^{\underline{k}} \end{aligned}$$

Euler's Summation Formula

Taylor's Theorem,

$$f(x+\epsilon) = f(x) + \frac{f'(x)}{1!}\epsilon + \frac{f''(x)}{2!}\epsilon^2 + ...$$

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Setting $\epsilon = 1$,

$$\begin{aligned} f(x+1) - f(x) = \Delta f(x) &= \frac{f'(x)}{1!} + \frac{f''(x)}{2!} + \frac{f'''(x)}{3!} + \ldots \\ &= \left(\frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \ldots\right) f(x) = (e^D - 1) f(x) \end{aligned}$$

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where B_k is the k^{th} Bernoulli number. Therefore,

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$$\sum = \frac{B_0}{D} + \frac{B_1}{1!} + \frac{B_2}{2!} D + \frac{B_3}{3!} D^2 + ... = \int + \sum_{k \geq 1} \frac{B_k}{k!} D^{k-1}$$

Euler's Summation Formula

Therefore,

$$\sum\nolimits_a^b f(x)\ \delta x = \int_a^b f(x)\ dx + \sum_{k\geq 1} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b$$

Euler's Summation Formula

Using our summation formula,

$$H_{n-1} = \sum\nolimits_1^n \frac{1}{x}\ \delta x = \int_1^n \frac{1}{x}\ dx + \sum_{k\geq 1} \frac{B_k}{k!} f^{(k-1)}(x) \Big|_1^n$$

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So

$$H_n - \ln n \approx -\frac{1}{n} + \sum_{k\geq 1} \frac{B_k}{k!}\left(f^{(k-1)}(n) - f^{(k-1)}(1)\right)$$

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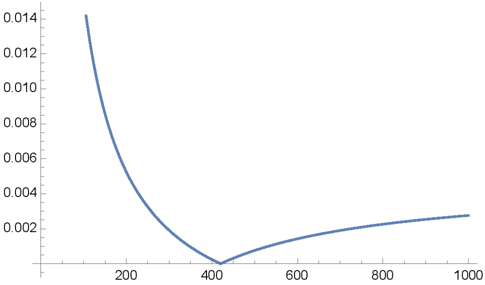
$$H_n - \ln n \approx -\frac{1}{n} + \sum_{k\geq 1} \frac{B_k}{k!} \left((-1)^{k+1}(k-1)!(n^{-k} - 1) \right)$$

$$= -\frac{1}{n} + \sum_{k\geq 1} (-1)^{k+1} \frac{B_k}{k} (n^{-k} - 1)$$

Euler's Summation Formula

$$\left|(H_{100} - \ln 100) - \left[\frac{1}{100} + \sum_{k=1}^{1000} \frac{B_k}{k!}\left(f^{(k-1)}(100) - f^{(k-1)}(1)\right)\right]\right| = 1.52 \times 10^{-2}$$



Question: Why does the error minimize around $n = 400$?