Chromatic Symmetric Functions

2025 Signature Work Conference and Exhibition

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Contents

Proper Colorings

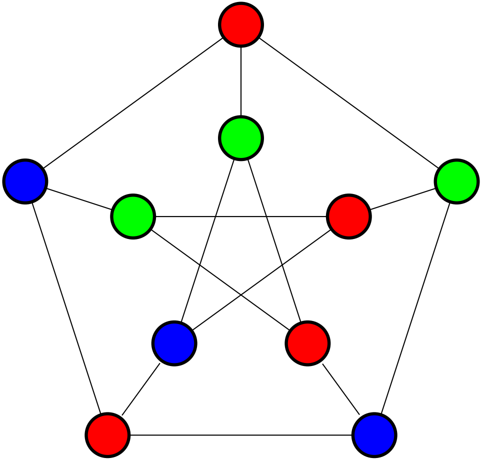

Figure: A proper 3-coloring of the Petersen graph.

Symmetric Polynomials

A polynomial is **symmetric** if it is unchanged under a *swapping* of its variables.

For example,

$$\begin{aligned} f(x,y,z) &= x^2y^2z + x^2z^2y + y^2z^2x \\ &= f(x,z,y) = f(y,x,z) = f(z,x,y) = f(y,z,x) = f(z,y,x) \end{aligned}$$

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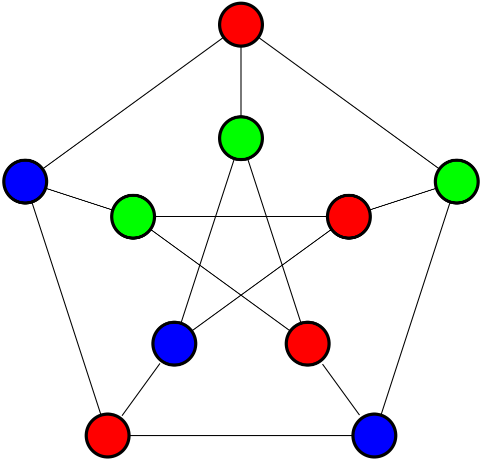
The set of symmetric functions homogeneous of degree n, Λ^n, is a vector space over $\mathbb{Q}$.

Chromatic Symmetric Function



$$f(\textcolor{yellow}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{red}{\bullet}) = \textcolor{yellow}{\bullet}^3 \textcolor{blue}{\bullet}^3 \textcolor{red}{\bullet}^4 \equiv \mathbf{x}^{\phi}$$

Chromatic Symmetric Function

Chromatic symmetric function (CSF) of a graph G,

$$X_G = \sum_{\phi \text{ proper}} \mathbf{x}^{\phi}$$

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Open question. Does the CSF tell tree graphs apart?

Chromatic Basis Theorem

Theorem. (Cho and Willenburg, 2008) Let G_k be connected with k vertices and $G_\lambda = G_{\lambda_1} + \ldots + G_{\lambda_{l(\lambda)}}$, then,

$$\{X_{G_\lambda} : \lambda \vdash n\}$$

is a $\mathbb{Q}$-basis for Λ^n.

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Example: Let S_k be the star graph with k vertices. If $\lambda = (5,4,4,2,1) \vdash 16$, then,

$$X_{S_\lambda} = X_{S_5} \cdot X_{S_4}^2 \cdot X_{S_2} \cdot X_{S_1}$$

Moreover,

$$X_{S_{k+1}} = \sum_{r=0}^{k} (-1)^r \binom{n}{r} p_{(r+1,1^{k-r})}$$

Chromatic Bases

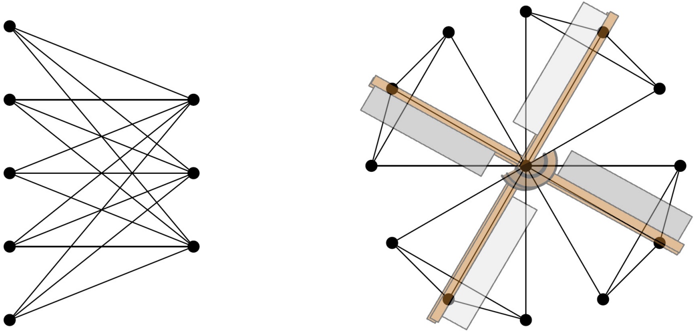

Figure: Complete bipartite graph $K_{5,3}$ (left), windmill graph $W_{4,4}$ (right).

Theorem. (Campbell, 2025) Let $K_{n,m}$ be the complete bipartite graph with $n+m$ vertices and $W_{k,r}$ be the windmill graph which is the composition of r copies of K_k. Then,

$$\text{(i)} \quad X_{K_{n,m}} = \sum_{\lambda \vdash (n+m)} \sum_{\substack{\mu \vdash n \\ \mu \subset \lambda}} \frac{\widetilde{\lambda}}{\widetilde{\mu} \cdot \widetilde{(\lambda - \mu)}} \frac{n! \cdot m!}{\lambda_1! \lambda_2! \ldots \lambda_{l(\lambda)}!} m_\lambda$$

$$\text{(ii)} \quad X_{W_{k,r}} = (k-1)!^r \sum_{\lambda \vdash r(k-1)+1} r_1 M_{(\lambda-(1)),((k-1)^r)} m_\lambda$$

where r_1 is the number of 1's in λ, and $M_{(\mu,((k-1)^r)}$ is the number of $l(\mu) \times r$ matrices with entires in $\{0,1\}$ such that there are exactly $(k-1)$ 1s in each column and μ_j 1s in the j^{th} row.

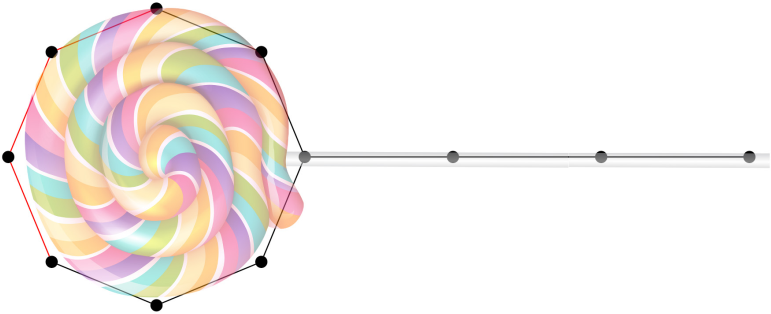

Figure: Lollipop graph $L_{11,4}$.

Chromatic Bases

Theorem. (Campbell, 2025) Let $L_{n,c}$ be the unique lollipop graph on n vertices with girth c, then,

$$X_{L_{n,c}} = \sum_{\lambda \vdash n} C_\lambda^{n,c} p_\lambda$$

where $C_\lambda^{n,c} = f(C_\lambda^{n-1,c}, C_\lambda^{n-2,c}, ..., C_\lambda^{c+1,c})$. In particular, we give an explicit expression for f and $C_\lambda^{c+1,c}$.

Reconstructing Spiders

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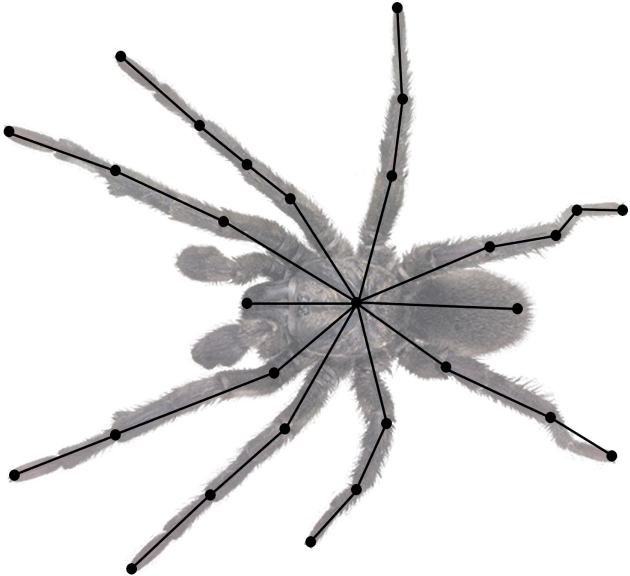
Reconstructing Spiders



Reconstructing Spiders

Theorem. (Martin, Morin, and Wager, 2008; Crew, 2022) Spiders are distinguished by their CSF.

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Theorem. (Campbell, 2025) Spiders can be *reconstructed* by their CSF.

Decomposing Trees

Theorem. (Orellana and Scott, 2014)

Decomposing Trees

Theorem. (Campbell, 2025) Let T_1, T_2 be two labeled trees. Then there is an algorithm which produces two sets of forests $\mathcal{F}_1, \mathcal{F}_2$, each with precisely $n-2$ edges, such that,

$$X_{T_1} = X_{T_2} + \sum_{\substack{F_1 \in \mathcal{F}_1 \\ F_2 \in \mathcal{F}_2}} X_{F_1} - X_{F_2}$$

References

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2. Jeremy L. Martin, Matthew Morin, and Jennifer D. Wagner. "On distinguishing trees by their chromatic symmetric functions". In: *Journal of Combinatorial Theory, Series A* 115 (2008), p. 103143.
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4. Rosa Orellana and Geoffrey Scott. "Graphs with equal chromatic symmetric functions". In: *Discrete Mathematics* 320 (2014), pp. 1–14.