Differentially Private Counting Queries on Approximate Shortest Paths

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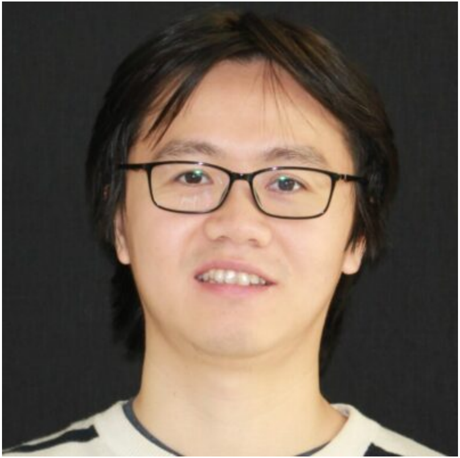
Contents

Acknowledgments

Joint work with **Dr. Chunjiang Zhu**, Assistant Professor, University of North Carolina at Greensboro.



Acknowledgments

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Counting Query Definition

A **counting query** over a path $P \subset E$ is the number $\sum_{e \in P} \phi(e)$

Motivation

Range counting – counting the number of points that are contained in a certain geometrically-defined range

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- *Edge weights* represent travel times along paths
- *Private edge attributes* represent number of patients in-transfer along paths

Differential Privacy Preliminaries

Neighboring Graphs

Two isomorphic graphs $G_1, G_2 = (V, E, \omega)$ with edge attribute functions $\phi_1, \phi_2 : E \to \mathbb{R}^+$ are said to be neighboring if

$$\sum_{e \in E} |\phi_1(e) - \phi_2(e)| \leq 1$$

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Sensitivity

The l_1 sensitivity of $\mathcal{A} : \mathcal{X} \to \mathbb{R}^D$ is defined as

$$\Delta_1(\mathcal{A}) \coloneqq \max_{X, X'} \|\mathcal{A}(X) - \mathcal{A}(X')\|_1$$

where X, X' are neighboring datasets.

Differential Privacy

Differential Privacy (DP) Definition

An algorithm $\mathcal{A} : \mathcal{X} \to \mathbb{R}^D$ is said to be (ε, δ)-differentially private if, for all outcomes $S \subseteq \mathbb{R}^D$ and neighboring datasets X, X',

$$\mathbb{P}[\mathcal{A}(X) \in S] \leq e^{\varepsilon} \cdot \mathbb{P}[\mathcal{A}(X') \in S] + \delta$$

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We call the case where $\delta = 0$ *pure* differential privacy and the case where $\delta > 0$ *approximate* differential privacy.

Results Overview

Citation	**Path System**	ε-**DP**	(ε, δ)-**DP**
Deng et al. (2023)	Shortest	$\widetilde{O}(n^{1/3})$	$\widetilde{O}(n^{1/4})$
Bodwin et al. (2024)	Shortest	–	$\widetilde{\Omega}(n^{1/4})$
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In general, let $\mathbf{T}$ be a t-collective tree spanner of G such that $|\mathbf{T}| = \eta_t$. There is a ε-DP algorithm for releasing the counting queries that is $\widetilde{O}(\eta_t)$-accurate and a (ε, δ)-DP algorithm that is $\widetilde{O}(\sqrt{\eta_t})$-accurate with probability $1 - \gamma$.

Differential Privacy Technology

Basic Composition

Let $\varepsilon, \delta \in [0, 1]$ and $k \in \mathbb{N}$. If we run k mechanisms where each mechanism is $(\varepsilon/k, \delta/k)$-DP, then the entire algorithm is (ε, δ)-DP.

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Laplace Mechanism

Given any function $f : \mathcal{X} \to \mathbb{R}^k$, the **Laplace mechanism** on input $X \in \mathcal{X}$ independently samples $Y_1, ..., Y_k$ according to $\text{Lap}(\Delta_1(f)/\varepsilon)$ and outputs,

$$\mathcal{M}_{f,\varepsilon}(X) = f(X) + (Y_1, ..., Y_k)$$

The Laplace mechanism is ε-differentially private.

Collective Tree Spanners

Collective Tree Spanner Definition

A collection of spanning trees $\mathbf{T}$ of G is said to be an α-collective tree spanner of G if for every $u, v \in V$, there is a tree $\mathcal{T} \in \mathbf{T}$ such that $d_{\mathcal{T}}(u, v) \leq \alpha \cdot d_G(u, v)$.

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Abraham et al. (2020). There is a polynomial time deterministic algorithm that finds a $(k \log \log (n))$-collective tree spanner with size $k \cdot n^{1/k}$.

A Recursive Tree Algorithm

Lemma 5. Let $\mathcal{T} = (V, E, \omega)$ be a tree rooted at $z \in V$ with $\varepsilon \in (0, 1]$ and $\gamma \in (0, 0.5]$. Then there is an ε-DP algorithm for releasing counting queries from the root to all other vertices on $\mathcal{T}$ that is $O(\log^{1.5}(n) \cdot \log(n/\gamma)/\varepsilon)$-accurate with probability $1 - \gamma$.

A Recursive Tree Algorithm

Algorithm 1 DPCQ on Rooted Trees

Input: Tree $\mathcal{T} = (V, E, \mathbf{w})$ rooted at $z \in V$ with edge attribute ϕ; parameter $\varepsilon \in (0, 1)$

Output: ε-DP approximate counting queries in $\mathcal{T}$, $\{\widetilde{\omega}(u, v)\}_{u,v \in V}$

1: Let z^* be the vertex in $\mathcal{T}$ such that the subtree rooted at z^* has more than $n/2$ vertices, but the subtree rooted at each of z^*'s children has at most $n/2$ vertices

2: Let $z_1, z_2, ..., z_\alpha$ be the children of z^*

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4: Sample $X \sim \text{Lap}(\log(n)/\varepsilon)$ and let $\omega(z^*, \mathcal{T}) = \omega(z, z^*) + X$

5: Let $\omega(z, \mathcal{T}) = 0$

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Remark 1. Every released counting query is the sum of at most $\log n$ estimates.

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Accuracy analysis. We apply the following lemma,

Laplace R.V. Concentration Bound

Let $X_1, \ldots, X_t$ be independent random variables distributed according to $\mathrm{Lap}(b)$, and let $X = X_1 + \ldots + X_t$. Then for all $\gamma \in (0, 1)$, with probability at least $1 - \gamma$ we have,

$$|X| < O(b\sqrt{t} \log(1/\gamma))$$

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The error in each estimate $\widetilde{\omega}(u, v)$ is the sum of at most $\log n$ i.i.d. Laplace R.V.s distributed according to $\mathrm{Lap}(\log(n)/\varepsilon)$, therefore by the previous bound,

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$$|\widetilde{\omega}(u, v) - \omega_T(u, v)| \leq O(\log^{1.5}(n) \cdot \log(1/\gamma)/\varepsilon)$$

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Privacy analysis. The estimates computed in each recursion are $(\varepsilon/\log(n))$-DP by the **Laplace Mechanism**.

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Privacy analysis. The estimates computed in each recursion are $(\varepsilon/\log(n))$-DP by the **Laplace Mechanism**. By **Basic Composition**, the entire algorithm is ε-DP.

Extension to General Trees

Algorithm 1 suffices to compute DPCQ for any tree.

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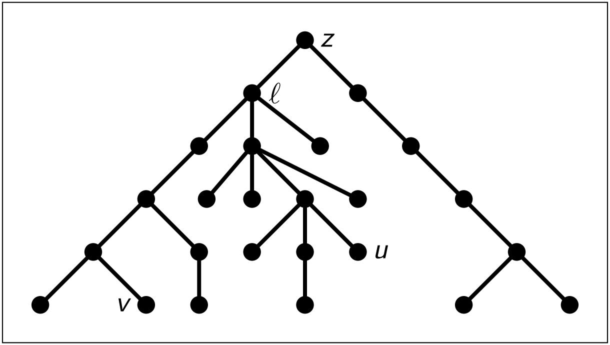
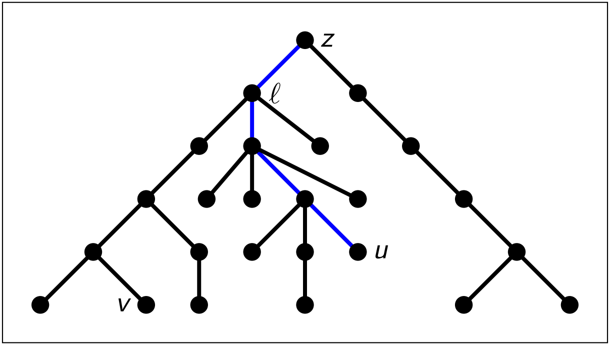


$$\widetilde{\omega}(u, v) =$$

Extension to General Trees

Algorithm 1 suffices to compute DPCQ for any tree. Arbitrarily pick a root z and let ℓ be the least common ancestor of $u, v \in V$.



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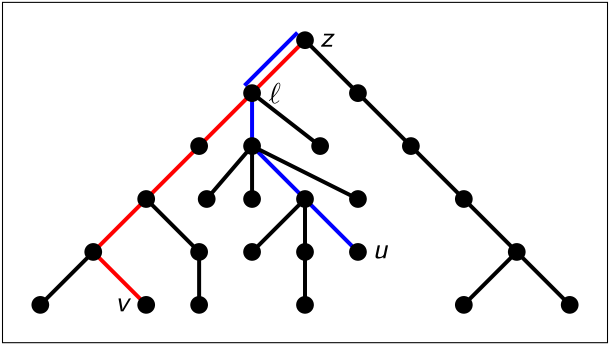

$$\widetilde{\omega}(u, v) = \color{blue}{\widetilde{\omega}(z, u)} + \color{red}{\widetilde{\omega}(z, v)}$$

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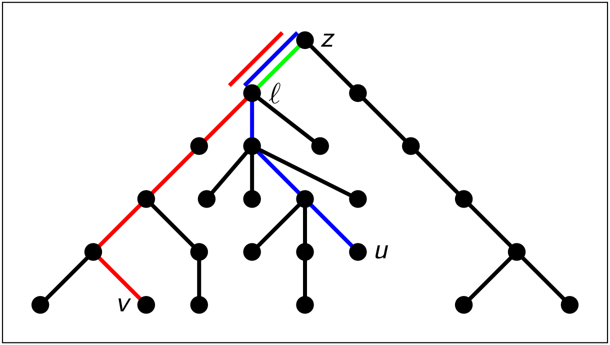

$$\widetilde{\omega}(u, v) = \widetilde{\omega}(z, u) + \widetilde{\omega}(z, v) - 2 \cdot \widetilde{\omega}(z, \ell)$$

A Generalization

Theorem 1. Let $\mathbf{T}$ be a t-collective tree spanner of G such that $|\mathbf{T}| = \eta_t$. For $\gamma \in (0, 0.5]$ and $\varepsilon \in (0, 1]$, there is an ε-DP algorithm for releasing the counting query between $u, v \in V$ on a t-approximate shortest path in G that is $O(\eta_t \cdot \log^{2.5}(n) \cdot \log(1/\gamma)/\varepsilon)$-accurate with probability $1 - \gamma$.

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Proof. We run the (ε/η_t)-DP mechanism given in **Algorithm 1** on each tree in $\mathbf{T}$. By **Basic Composition** and a union bound, we can release the estimates over all of $\mathbf{T}$ with

$$O(\eta_t \cdot \log^{2.5}(n) \cdot \log(1/\gamma)/\varepsilon)$$

error with probability $1 - \gamma$. □

Collective Tree Spanners

Assuming the Erdös girth conjecture, any $(2k-1)$ spanner must have at least $\Omega(n^{1+(1/k)})$ edges, and this bound is tight.

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Open question 1 – Can we construct a collective tree spanner that has a size/stretch tradeoff that is closer to being optimal?

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Open question 2 – Close the gap between these upper- and lower-bounds.

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