Benford's Law

Discrete Mathematics Seminar 2023

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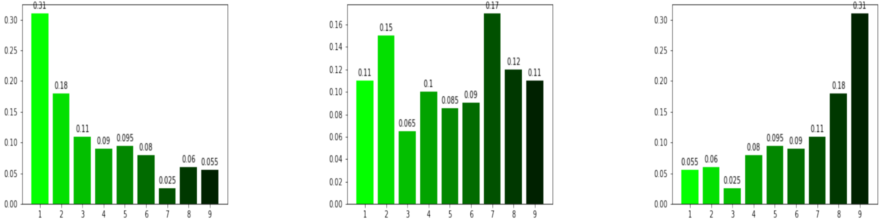
Outline

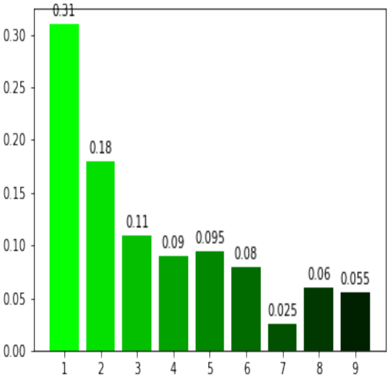
Open questions will be in orange!

Country Populations



Frequency of leading digits of population by country (2020).

Country Populations



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Fibonacci Numbers

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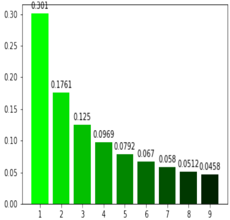
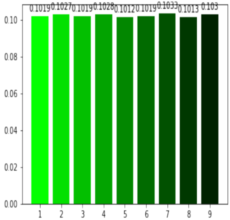
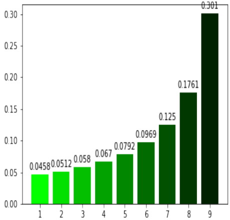
Which graph is correct?

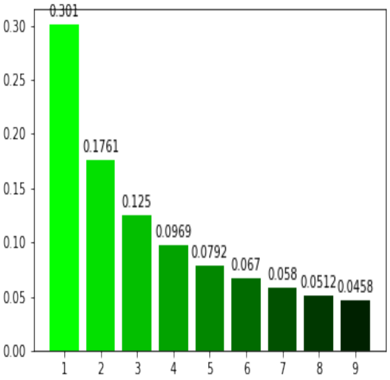






Frequency of leading digits of first 50,000 Fibonacci numbers.

Fibonacci Numbers



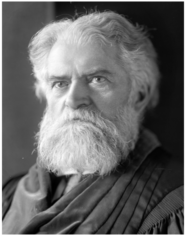
Frequency of leading digits of first 50,000 Fibonacci numbers.

Benford's Law Visualization

Digit	Probability $P(d)$	Relative Size of $P(d)$
1	0.301	
2	0.176	
3	0.125	
4	0.097	
5	0.079	
6	0.067	
7	0.058	
8	0.051	
9	0.046	

History of Benford's Law

That the ten digits do not occur with equal frequency must be evident to any one making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones. —*Simon Newcomb (1881)*



Simon Newcomb, 1905

History of Benford's Law

- Benford's Law was rediscovered by physicist Frank Benford in 1938.
- Compiled over 20 tables containing over 20,000 data points supporting the law

TABLE I

PERCENTAGE OF TIMES THE NATURAL NUMBERS 1 TO 9 ARE USED AS FIRST DIGITS IN NUMBERS, AS DETERMINED BY 20,229 OBSERVATIONS

Group	Title	First Digit									Count
		1	2	3	4	5	6	7	8	9	
A	Rivers, Area	31.0	16.4	10.7	11.3	7.2	8.6	5.5	4.2	5.1	335
B	Population	33.9	20.4	14.2	8.1	7.2	6.2	4.1	3.7	2.2	3259
C	Constants	41.3	14.4	4.8	8.6	10.6	5.8	1.0	2.9	10.6	104
D	Newspapers	30.0	18.0	12.0	10.0	8.0	6.0	6.0	5.0	5.0	100
E	Spec. Heat	24.0	18.4	16.2	14.6	10.6	4.1	3.2	4.8	4.1	1389
F	Pressure	29.6	18.3	12.8	9.8	8.3	6.4	5.7	4.4	4.7	703
G	H.P. Lost	30.0	18.4	11.9	10.8	8.1	7.0	5.1	5.1	3.6	690
H	Mol. Wgt.	26.7	25.2	15.4	10.8	6.7	5.1	4.1	2.8	3.2	1800
I	Drainage	27.1	23.9	13.8	12.6	8.2	5.0	5.0	2.5	1.9	159
J	Atomic Wgt.	47.2	18.7	5.5	4.4	6.6	4.4	3.3	4.4	5.5	91
K	$n^{-1}, \sqrt{n}, \cdots$	25.7	20.3	9.7	6.8	6.6	6.8	7.2	8.0	8.9	5000
L	Design	26.8	14.8	14.3	7.5	8.3	8.4	7.0	7.3	5.6	560
M	*Digest*	33.4	18.5	12.4	7.5	7.1	6.5	5.5	4.9	4.2	308
N	Cost Data	32.4	18.8	10.1	10.1	9.8	5.5	4.7	5.5	3.1	741
O	X-Ray Volts	27.9	17.5	14.4	9.0	8.1	7.4	5.1	5.8	4.8	707
P	Am. League	32.7	17.6	12.6	9.8	7.4	6.4	4.9	5.6	3.0	1458
Q	Black Body	31.0	17.3	14.1	8.7	6.6	7.0	5.2	4.7	5.4	1165
R	Addresses	28.9	19.2	12.6	8.8	8.5	6.4	5.6	5.0	5.0	342
S	$n^1, n^2 \cdots n!$	25.3	16.0	12.0	10.0	8.5	8.8	6.8	7.1	5.5	900
T	Death Rate	27.0	18.6	15.7	9.4	6.7	6.5	7.2	4.8	4.1	418
Average		30.6	18.5	12.4	9.4	8.0	6.4	5.1	4.9	4.7	1011
Probable Error		±0.8	±0.4	±0.4	±0.3	±0.2	±0.2	±0.2	±0.2	±0.3	—

Frank Benford's original data supporting Benford's Law (1938)

Notation

Let $D_n(x)$ be the n^{th} significant decimal digit

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- $D_1(\pi) = 3$, $D_2(\pi) = 1$, $D_3(\pi) = 4$
- $D_n(300) = D_n(3) = D_n(0.003)$

What is Benford's Law

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$$Prob(D_1 = d_1, D_2 = d_2, ..., D_m = d_m) = \log_{10}(1 + (\textstyle\sum_{j=1}^{m} 10^{m-j} d_j)^{-1})$$

Example

Pick any number from a distribution that follows Benford's Law.

What's the probability that the first five digits are the same as π?

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$\text{Prob}(\text{D}_1 = 3, D_2 = 1, D_3 = 4, D_4 = 1, D_5 = 5) = \log_{10}(1 + \frac{1}{31415})$

$= \log_{10}(\frac{31416}{31415}) \approx 0.0000138$

Suprising Result?

$$\text{Prob}(D_2 = 1) = \sum_{j=1}^{9} \log_{10}(1 + \frac{1}{10j+1}) = \log_{10}(\frac{6029312}{4638501}) \approx 0.1138$$

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Conclusion: Significant digits are **dependent**.

Significand

Another useful concept when taking about Benford's Law is the **significand**, also called the *mantissa*.

The significand of a number, call it $S(x)$, is its coeffcient when expressed in "scientific" (floating-point) notation.

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$P = 6.626 \times 10^{-34}$ (Plank Constant)
$S(P) = 6.626$

Significand Function

Explicity, the base-10 significand function $S : \mathbb{R} \to [1, 10)$ is given by,

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Using the significand we can state Benford's Law in a new (and super concise) way:

Benford's Law

$$Prob(S \leq t) = \log(t),\ t \in [1, 10)$$

σ-Algebras

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Example: The *power set* of Ω, which is the set containing all possible subsets of Ω, is the largest possible σ-algebra on Ω.

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$\sigma(f)$ is the *smallest* σ-algebra on Ω that contains all sets of the form $\{\omega \in \Omega : a \leq f(\omega) \leq b\}$ for every $a, b \in \mathbb{R}$.

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Whereas, for example, the interval $[1, 2]$ does not belong to $\mathcal{S}$.

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Where $P(d)$ is the probability of picking a number from distribution $f(x)$ beginning with d.

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Make the following substitutions:

$$t = d \cdot 10^n; \;\; dn = \frac{dt}{tln(10)}$$
$$x = ty; \;\; dx = tdy$$

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Giving

$$P(d) \approx \int_0^{\infty} \int_1^{1+\frac{1}{d}} f(ty)t\,dy \cdot \frac{dt}{t \ln 10} = \frac{1}{\ln 10} \int_0^{\infty} dt \int_1^{1+\frac{1}{d}} f(ty)\,dy$$

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$$= \frac{1}{\ln 10} \int\limits_1^{1+\frac{1}{d}} \frac{1}{y} dy \int\limits_0^{\infty} f(x)\, dx = \frac{\ln 1+\frac{1}{d}}{\ln 10} \int\limits_{\Omega} f(x)\, dx = \frac{\ln 1+\frac{1}{d}}{\ln 10}$$

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By the change-of-base rule for logarithms, we are left with,

Mathematical Derivation

Let $f(ty) = \phi(y,t)$. By Fubini's Theorem,

$$P(d) \approx \frac{1}{ln10} \int\limits_0^\infty dt \int\limits_1^{1+\frac{1}{d}} \phi(y,t)\,dy = \frac{1}{ln10} \int\limits_1^{1+\frac{1}{d}} dy \int\limits_0^\infty \phi(y,t)\,dt$$

$$= \frac{1}{\ln 10} \int\limits_1^{1+\frac{1}{d}} \frac{1}{y} dy \int\limits_0^\infty f(x)\,dx = \frac{\ln 1+\frac{1}{d}}{\ln 10} \int\limits_\Omega f(x)\,dx = \frac{\ln 1+\frac{1}{d}}{\ln 10}$$

By the change-of-base rule for logarithms, we are left with,

Benford's Law

$$P(d) = \log_{10}\left(1 + \frac{1}{d}\right)$$

Benford's Law Derivation

Thus, even though many common sequences... do not follow Benford's Law, those that do are so ubiquitous that many authors have assumed that a simple explanation must exist... [however], there does not appear to be a simple derivation of Benford's Law that both offers a "correct explanation" and... provide(s) insight. —*Arno Berger (2011)*

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I think in statistics we need derivations, not proofs. That is, lines of reasoning from some assumptions to a formula, or a procedure, which may or may not have certain properties in a given context, but which, all going well, might provide some insight. —*Terry Speed (2009)*

Questions you may have...

- How do we derive the distribution function for Benford's Law?
- **Which distributions of numbers follow Benford's Law?**

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Benford Sequence

$$\lim_{N\to+\infty} \frac{\#\{1\leq n\leq N : S(x_n)\leq t\}}{N} = \log t, \quad \text{for all } t \in [1, 10)$$

Where $\#\{\cdot\}$ denotes the number of elements in the set.

Example (Natural Numbers)

Is the sequence of natural numbers $(x_n) = n$ Benford?

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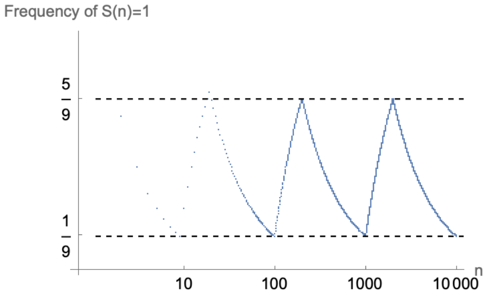

Example (Natural Numbers)

As we might expect, we see that,

$$\liminf N \to +\infty\left(\frac{\#\{N\in[1,n]:S(N)=1\}}{n}\right) = \frac{1}{9}$$

and,

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So the limit does not exist, and $(x_n) = n$ is not Benford!

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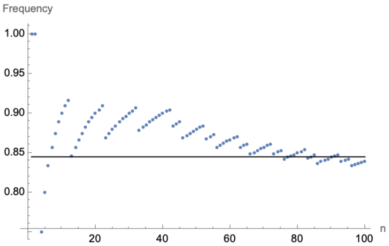

Plot of n vs. $\frac{1}{n} \cdot \#\{N \in [1, n] : S(2^N) \leq 7\}$ with line $y = \log(7)$ shown.

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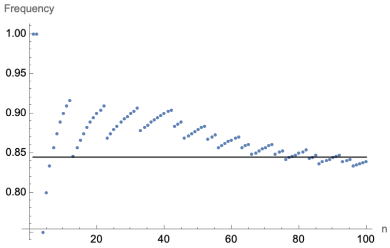

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$(x_n) = {2^n}_{\text{base } 2} = 1, 10, 100, 1000, 10000, ...$

$Prob({D_2}^{(2)} = 0) = 1 - Prob({D_2}^{(2)} = 1) = \log_2(3) - 1 > \frac{1}{2}$

So $(x_n) = {2^n}_{\text{base } 2}$ is not Benford!

Other Benford Sequences

- The Fibonacci Sequence

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- $(f^n(x_0))$ where $f(x) = ax^b$ with $a > 0, b > 1$
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- (θ^n) for any irrational θ.
- Prime numbers
  - "Logarithmic Benford"
  - Logarithmic density of $\{n \in \mathbb{N} : S(x_n) \leq t\} = \log(t)$

Application to Newton's Method

Newton's Method is used to approximate the roots of real-valued functions using the function,

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It can be shown that for x_0 sufficiently close to a root x' (i.e. $g(x') = 0$), that

$$\lim_{n\to\infty}({N_g}^n(x_0)) = x'$$

Application to Newton's Method

Theorem 3.1

Let the function $g : I \to \mathbb{R}$ be real-analytic with $g(x') = 0$, and assume that g is not linear.

(i) If x' is a simple root (multiplicity 1), then $(x_n - x')$ and $(x_{n+1} - x_n)$ are both Benford for almost all x_0 in a neighborhood of x'.

(ii) If x' has multiplicity ≥ 2, then $(x_n - x')$ and $(x_{n+1} - x_n)$ are Benford for all $x_0 \neq x'$ sufficiently close to x'.

Application to Newton's Method

Example: Let $g(x) = e^x - 2$, then g has a root at $x' = \ln(2)$ and $N_g(x) = x - 1 + 2e^{-x}$. By the above theorem, the sequences $(x_n - x')$ and $(x_{n+1} - x_n)$ are both Benford for almost all x_0 near x'.

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Why it matters?: In computer algorithms, roundoff errors are inevitable. In computer implementations of Newton's Method, there is normally an assumption of uniformly distributed fraction parts. Such an assumption would lead to an underestimate in the average relative round-off error in the above case.

Application to Newton's Method

"[I]n order to analyze the average behavior of floating-point arithmetic algorithms, we need some statistical information that allows us to determine how often various cases arise... [If, for example, the] leading digits tend to be small [, that] makes the most obvious techniques of "average error" estimation for floating-point calculations invalid. The relative error due to rounding is usually... more than expected. —Donald Knuth, The Art of Computer Programming (1968)

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Question: Can Benford's Law improve current roundoff error approximation techniques in floating-point arithmetic?

Benford Random Variables

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- $\mathbb{P}(D_1(X) = 3, D_2(X) = 1, D_3(X) = 4) = \log(\frac{315}{314})$

$\mathbb{N}$-Valued Random Variables

One might consider classifying $\mathbb{N}$-valued random variables (i.e. $\mathbb{P}(X \in \mathbb{N}) = 1$) as *Benford on* N if

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However, no such random variable exists!

Which Distributions are Benford?

None of the standard continuous probability distributions (e.g., uniform, exponential, normal, etc.) are Benford, however their deviation from Benford's law can be quantified using the metric

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$$\Delta_\infty := 100 \cdot \sup_{1 \leq t < 10} |F_{S(X)}(t) - \log(t)|$$

Where $\Delta_\infty = 0$ if and only if X is Benford and $\Delta_\infty = 100$ if and only if $\mathbb{P}(S(X) = 1) = 1$

Example: Exponential Distribution

Consider the exponential distribution centered a 1 with cumulative distribution given by:

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$$\mathbb{P}(D_1(X) = 1) > (e^{\frac{1}{10}} - e^{\frac{-2}{10}}) + (e^{-1} - e^{-2}) + (e^{-10} - e^{-20}) \approx 0.3186 > \log(2)$$

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$\Delta_\infty = 3.05$ i.e. $|\mathbb{P}(S(X) \leq t) - \log(t)|$ is small for $t \in [1, 10)$.

Other Common Distributions

Here is a table of other common distributions and how closely they follow Benford's Law:

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Distributions	Δ_∞
Uniform [0,1]	26.88
Exponential(1)	3.05
Pareto(1)	26.88
Arcsin	28.77
Standard Normal	6.05

Note on Uniform Distribution

Theorem 4.1

For every uniformly distributed positive random variable X,

$$\max_{1 \leq t < 10} |F_{S(X)}(t) - \log(t)| \geq \frac{1}{18} + \frac{1}{2}(\log(9) - \log(e) + \log\log(e)) \approx 0.1344$$

And this bound is *sharp*.

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Fallacy: Regularity and large spread implies Benford's Law.

Now, this claim is clearly false. No matter how large the spread, if data follows a uniform distribution then it does not conform to Benford's Law.

Uniform Distribution Modulo 1

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Uniform Distribution Modulo 1 (Random Variables)

This definition has a natural extension to random variables, namely,

Uniform Distribution Modulo 1 (Random Variables)

A random variable (r.v.) X on a probability space $(\Omega, \sigma, \mathbb{P})$ is *u.d.* mod 1 if

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Connection to Benford's Law

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Importance: Theorem 5.1 is one of the main tools in the theory of Benford's law because it allows application of the powerful theory of uniform distribution modulo one.

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The proofs for sequences are completely analagous.

Applications

Proposition: Let $(x_n) = (x_1, x_2, ...)$ be a sequence of real numbers.

If $\lim_{n\to\infty}(x_{n+1} - x_n) = \theta$ for some irrational θ, then (x_n) is *u.d.* mod 1.

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Example: Consider the family of sequences $(d_n) = (n\log(\alpha))$. If $\log(\alpha)$ is irrational, i.e. $\alpha = 2$ or $\alpha = \pi$, then by the above proposition (d_n) is *u.d.* mod 1.

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It is easy to show through this method that, for instance, θ^n is Benford for any irrational θ.

US Taxpayer Records

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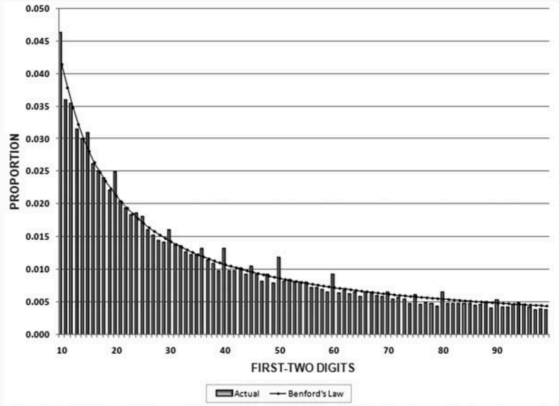
Note that we can use the general Benford's Law given by

$$Prob(D_1 = d_1, D_2 = d_2, ..., D_m = d_m) = \log_{10}(1 + (\textstyle\sum_{j=1}^{m} 10^{m-j} d_j)^{-1})$$

to compute the **joint probability** for the of the first *n* digits occurring.

US Taxpayer Records



Frequency of first two digits: Dividend income declared

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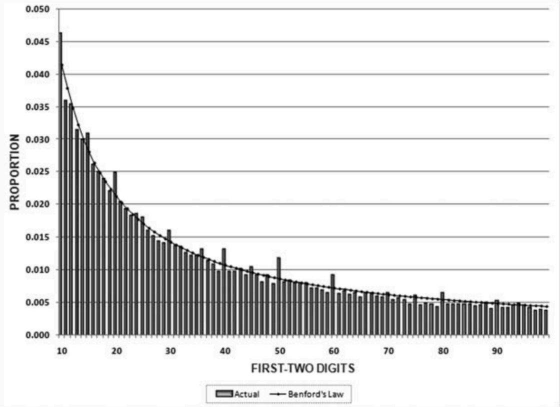

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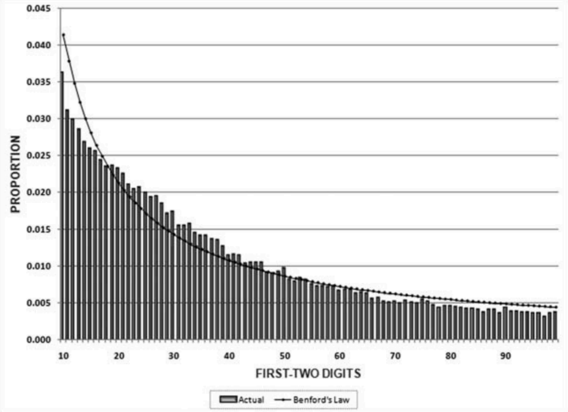
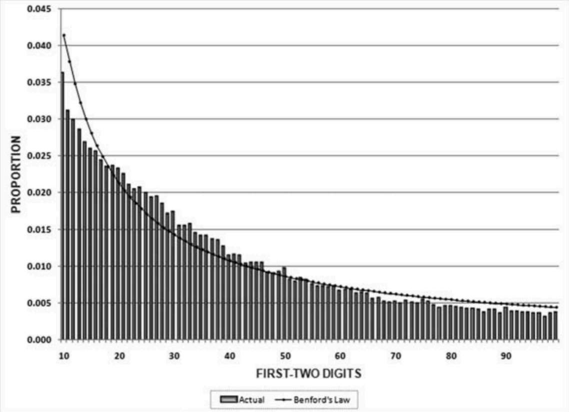
Question: Why are there spikes at multiples of 10?

US Taxpayer Records



Frequency of first two digits: Interest expense claimed

US Taxpayer Records



Frequency of first two digits: Interest expense claimed

Question: Why are the higher values supressed here?

2020 US Presidential Election

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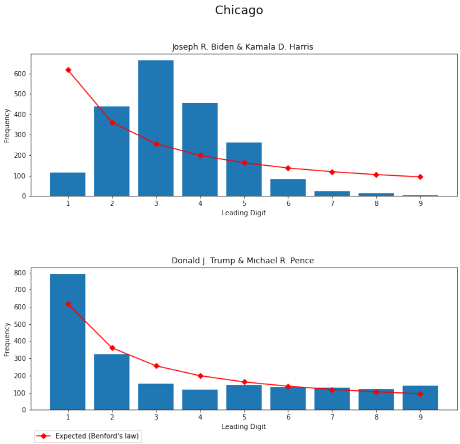
The city of Chicago has 2,069 precincts which report election data. Each precinct is roughly the same size, with the smallest reporting 39 votes, and the biggest 1655, with an average of 516 and a standard deviation of 173.

2020 US Presidential Election



Plots of 2020 Chicago presidential election data by candiadate for 2,069 precincts with the predicted values by Benford's Law shown.

2020 US Presidential Election

[I]f a competitive two candidate race occurs in districts whose magnitude varies between 100 and 1000, the modal first digit for each candidate's vote will not be 1 or 2 but rather 4, 5, or 6. — Henry E. Brady (2005)

References

Berger, A., Hill, T.P. (2015). An Introduction to Benford's Law. Princeton University Press.

Newcomb, S. (1881). Note on the Frequency of Use of the Different Digits in Natural Numbers. American Journal of Mathematics, 4(1), 39-40. http://links.jstor.org/sici?sici=0002-9327%281881%294%3A1%3C39%3ANOTFOU%3E2.0.CO%3B2-V

Benford, F. (1938). The Law of Anomalous Numbers. Proceedings of the American Philosophical Society, 78(4), 551-572. https://www.jstor.org/stable/984802

Wang, L., Ma, B. Q. (2023). A concise proof of Benford's Law. Fundamental Research, 1-5. https://doi.org/10.1016/j.fmre.2023.01.002

Berger, A., Hill, T.P. Benford's Law Strikes Back: No Simple Explanation in Sight for Mathematical Gem. Math Intelligencer 33, 85–91 (2011). https://doi.org/10.1007/s00283-010-9182-3

Speed, T. (2009), ''You Want Proof?'', Bull. Inst. Math. Statistics 38, p 11.

Nigiri, M. J. (2009). Benford's Law (pp. 283-285). John Wiley Sons, Inc.

[Stand-Up Maths]. (2020, November 11). Why do Biden's votes not follow Benford's Law? [Video]. Youtube.com. https://www.youtube.com/watch?v=etx0k1nLn78ab$_channel = Stand - upMaths$

Deckert, J., Myagkov, M., Ordeshook, P. (2011). Benford's Law and the Detection of Election Fraud. Political Analysis, 19(3), 245-268. doi:10.1093/pan/mpr014