# Expressions for the Chromatic Symmetric Function of Complete Bipartite, Windmill, and Lollipop Graphs

Jesse Campbell\*

#### Abstract

The chromatic symmetric function (CSF) is a symmetric function associated to a graph, defined as a sum over its proper colorings. One interesting result about the CSF that has inspired some recent work is forming a CSF basis for the set of symmetric functions. Surprisingly, any arbitrary collection of connected graphs, so long as there is exactly one graph of each vertex size up to n, can be used to form an algebraically independent basis for the set of symmetric functions homogeneous of degree n. This has inspired the work of finding simple expressions for the CSF of infinite families of connected graphs in-terms of known symmetric function bases. In this note, we give simple expressions for the complete bipartite and windmill graphs in the monomial symmetric function basis. Moreover, we give an explicit expression for lollipop graphs in the power sum symmetric function basis.

#### 1 Introduction

The chromatic symmetric function (CSF) was introduced by Richard Stanley in his 1995 seminal work, A Symmetric Function Generalization of the Chromatic Polynomial of a Graph [5]. Given a graph G = (V, E), the CSF of G is denoted as  $X_G$  and is defined as a sum over all proper colorings  $\phi : V \to \mathbb{N}$  of G, and is defined by,

$$X_G = \sum_{\phi} \mathbf{x}^{\phi} = \sum_{\phi} \prod_{v \in V} x_{\phi(v)}$$

Based on this definition, it is not hard to see the following result.

**Lemma 1.1.** Let  $G = (V_1, E_1)$  and  $H = (V_2, E_2)$  be graphs, and let G + H denote the disjoint union of G and H, then,

$$X_{G+H} = X_G \cdot X_H$$

In the same paper, Stanley also gave combinatorial interpretations of the CSF in the monomial and power sum symmetric function bases. Firstly, we state the result for the monomial basis.

**Lemma 1.2** ([5]). Let  $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n = \#V$  and  $a_{\lambda}$  be the number of partitions of V into components of size  $\lambda_1, \lambda_2, ..., \lambda_k$  such that there are no edges between any two vertices in the same component. Then,

$$X_G = \sum_{\lambda \vdash n} a_\lambda \widetilde{m}_\lambda$$

<sup>\*</sup>Duke Kunshan University (jesse.campbell@duke.edu)

Before we state the result for the power sum basis, as a bit of notation, if G = (V, E) is a graph and  $S \subset E$ , then we denote by G(S) = (V, S) the subgraph of G with edge set S. Moreover, we denote by  $\lambda_G(S)$  the partition of n = |V| formed by sorting the number of vertices in each connected component of G(S) in weakly decreasing order. Where G is obvious, we omit from the subscript and simply write  $\lambda(S)$ .

Lemma 1.3 ([5]).

$$X_G = \sum_{S \subset E} (-1)^{|S|} p_{\lambda(S)}$$

One interesting result about the chromatic symmetric function that has inspired much recent work is forming a CSF basis for the set of symmetric functions. Surprisingly, we have the following lemma of Cho and Willigenburg.

**Lemma 1.4** ([1]). Let  $\{G_k\}_{k\geq 1}$  be a set of connected graphs such that  $G_k$  has k vertices for each  $k \geq 1$ . Moreover, define  $G_{\lambda} = G_{\lambda_1} + G_{\lambda_2} + \ldots + G_{\lambda_{l(\lambda)}}$  where "+" is the disjoint union of graphs, then  $\{X_{G_{\lambda}} : \lambda \vdash n\}$  is a Q-basis for  $\Lambda^n$ .

Furthermore, the set  $\{X_{G_k}\}_{k\geq 1}$  is algebraically independent, like the set  $\{e_k\}_{k\geq 1}$ , such that one may even consider calling the above lemma the *Chromatic Version* of the Fundamental Theorem of Symmetric Functions. Previous work such as [1, 3] has examined the bases formed by CSFs of families of graphs with simple structures, such as paths and stars. In particular, the main contribution of [1] was giving explicit expansions of several simple families of graphs in known symmetric function bases in order to better understand chromatic bases. Their results are summarized in the following lemma.

**Lemma 1.5** ([1]). Let  $K_n$  be the complete graph,  $S_n$  be the star graph,  $P_n$  be the path graph, and  $C_n$  be the cycle graph, all on n vertices. Then,

(i) 
$$X_{K_n} = n!e_n$$
  
(ii)  $X_{S_{n+1}} = \sum_{r=0}^n (-1)^r \binom{n}{r} p_{(r+1,1^{n-r})}$   
(iii)  $X_{P_n} = \sum_{\lambda = (n^{r_n}, \dots, 1^{r_1}) \vdash n} (-1)^{n - \sum_{i=1}^n r_i} \frac{(\sum_{i=1}^n r_i)!}{\prod_{i=1}^n (r_i)!} p_\lambda$   
(iv)  $X_{C_n} = \sum_{\lambda = (n^{r_n}, \dots, 1^{r_1}) \vdash n} (-1)^{n - \sum_{i=1}^n r_i} \frac{(\sum_{i=1}^n r_i)!}{\prod_{i=1}^n (r_i)!} \left(1 + \sum_{j=2}^n (j-1) \frac{r_j}{\sum_{i=1}^n r_i}\right) p_\lambda + (-1)^n p_n$ 

*Proof.* See **Theorem 8** in [1], pg. 4.

We extend the results of [1] by giving explicit formulas for the chromatic symmetric function of the complete bipartite graph  $K_{n,m}$  and windmill graph  $W_{k,r}$  in the monomial symmetric function basis, and the lollipop graph  $L_{n,c}$  in the power sum basis. The complete bipartite graphs  $K_{n,m}$  are one of the most famously-studied families of graphs, which consist of a sets of n independent vertices and m independent vertices. Then, every possible edge between the two sets is in the edge set of  $K_{n,m}$ . The windmill graphs are a generalization of friendship graphs. Specifically,  $W_{k,r}$  is formed by taking r independent copies of the complete graph  $K_k$ , and adjoining the graphs together at one vertex. See Figure 1 for an example of a complete bipartite and windmill graph.

### 2 Expressions for Bipartite, Windmill, and Lollipop Graphs

Before we state the first theorem which gives expressions for complete bipartite graphs and windmill graphs in the monomial basis, we state the following general result about change-of-basis between monomial and elementary symmetric functions.

**Lemma 2.1** ([6]). Let  $\lambda = (\lambda_1, ..., \lambda_k), \mu = (\mu_1, ..., \mu_\ell) \vdash n$  be partitions and  $M_{\lambda,\mu}$  the number of 0, 1-matrices with row sums  $\lambda_i$  and column sums  $\mu_j$ , then,

$$e_{\mu} = \sum_{\lambda \vdash n} M_{\lambda,\mu} m_{\lambda}$$

Moreover, for simplicity, if  $\lambda = (n^{r_n}, (n-1)^{r_{n-1}}, ..., 2^{r_n}, 1^{r_1})$  where each  $r_i \ge 0$  for  $1 \le i \le n$ , we define  $\lambda = r_1! r_2! ... r_n!$ . Now, we go on to state our first main result.

**Theorem 2.1.** Let  $K_{n,m}$  be the complete bipartite graph with n+m vertices and  $W_{k,r}$  be the windmill graph which is the composition of r copies of  $K_k$ . Then,

(i) 
$$X_{K_{n,m}} = \sum_{\lambda \vdash (n+m)} \sum_{\substack{\mu \vdash n \\ \mu \subset \lambda}} \frac{\widetilde{\lambda}}{\widetilde{\mu} \cdot (\widetilde{\lambda - \mu})} \frac{n! \cdot m!}{\lambda_1! \lambda_2! \dots \lambda_{l(\lambda)}!} m_{\lambda}$$
  
(ii) 
$$X_{W_{k,r}} = (k-1)!^r \sum_{\lambda \vdash r(k-1)+1} r_1 M_{(\lambda - (1)), ((k-1)^r)} m_{\lambda}$$

where  $r_1$  is the number of 1's in  $\lambda$ , and  $M_{(\mu,((k-1)^r)}$  is the number of  $l(\mu) \times r$  matrices with entires in  $\{0,1\}$  such that there are exactly (k-1) 1s in each column and  $\mu_j$  1s in the  $j^{th}$  row.

Proof. (i) By **Lemma 1.2**, for each partition  $\lambda \vdash n+m$  the coefficient of  $\widetilde{m}_{\lambda}$  in  $X_{K_{n,m}}$  is the number of partitions of V into independent sets of type  $\lambda$ . Let  $V_1$  be the set of n vertices in  $K_{n,m}$  with no edges between them. Likewise, let  $V_2$  be the set of m vertices in  $K_{n,m}$  with no edges between then. Since for every  $u \in V_1$  and  $v \in V_2$ ,  $(u, v) \in E(K_{n,m})$ , then any independent set in  $K_{n,m}$  must be a subset of either  $V_1$  or  $V_2$ . We first pick the sizes of the independent sets in  $V_1$ , represented by  $\mu = (\mu_1, ..., \mu_k) \subset \lambda$ . Suppose that we have not chosen any independent sets from  $V_1$ , then there are  $\binom{n}{\mu_1}$  ways to choose an independent set of size  $\mu_1$ , since each vertex in  $V_1$  in independent set of size  $\mu_2$  from the remaining  $n - \mu_1$  vertices in  $V_1$  after the first independent set has already been chosen. Continuing in this way, there are,

$$\prod_{j_1}^{l(\mu)} \binom{n - \sum_{\ell=1}^{j_1 - 1} \mu_\ell}{\mu_{j_1 + 1}} = \frac{n!}{\mu_1!(n - \mu_1)!} \frac{(n - \mu_1)!}{\mu_2!(n - \mu_1 - \mu_2)!} \cdots \frac{(n - \sum_{\ell=1}^{j_1 - 1} \mu_\ell)!}{\mu_{l(\mu)}!(n - \sum_{\ell=1}^{l(\mu)} \mu_\ell)!} = \frac{n!}{\mu_1!\mu_2!\dots\mu_{l(\mu)}!}$$

ways to choose independent sets with sizes corresponding to  $\mu$  from the set  $V_1$  where each independent set is distinct.<sup>1</sup> It is not hard to see that, by the same logic, there are,

$$\prod_{j_2=1}^{l(\lambda)-l(\mu)} \binom{m - \sum_{\ell=1}^{j_2-1} (\lambda - \mu)_{\ell}}{(\lambda - \mu)_{j_2+1}} = \frac{m!}{(\lambda - \mu)_1! (\lambda - \mu)_2! \dots (\lambda - \mu)_{l((\lambda - \mu))}!}$$

ways to choose independent sets with sizes corresponding to  $\lambda - \mu$  from  $V_2$  where each independent set is distinct. Lastly, we must account for the overcounting in the choice of independent sets. Namely, if there are multiplicities in the partition  $\mu$  or  $(\lambda - \mu)$ , then we are overcounting the choices of independent sets with the size of these repeated numbers as the order in which they are selected does not matter. To account for this, we divide each product in the sum by  $1/\tilde{\mu}$  and  $1/(\tilde{\lambda} - \mu)$ , respectively.

<sup>&</sup>lt;sup>1</sup>This is equivalent to the counting performed by "multinomial coefficients", which are not used here for improved clarity of the result.



Figure 1: Complete bipartite graph  $K_{5,3}$  (left), windmill graph  $W_{4,4}$  (right)

(ii) We first note that there are no nontrivial independent sets (those containing more than one vertex) containing the center vertex (where all of the complete graphs are adjoined) in  $W_{k,r}$ , since it is connected to every other vertex in  $W_{k,r}$ . Hence, for any partition  $\lambda = ((\#V)^{r_{\#V}}, ..., 1^{r_1}) \vdash \#V$  such that  $r_1 = 0$ , we must have that the coefficient of  $m_{\lambda}$  in  $X_{W_{k,r}}$  is. 0. Therefore, we restrict ourselves to partitions with  $r_1 > 0$ .

Assuming that the central vertex is in an independent set of size 1, we wish to find the number of ways to groups the remaining vertices, which can be thought-of as r independent copies of  $K_{k-1}$ into independent sets of size  $\lambda - (1)$ . We call the graph of  $W_{k,r}$  with the central vertex and all of its adjacent edges removed  $W'_{k,r}$ . Moreover, by (i) in **Lemma 1.5**,

$$\begin{aligned} X_{W'_{k,r}} &= (X_{K_{k-1}})^r \\ &= ((k-1)!e_{k-1})^r \\ &= (k-1)!^r e_{((k-1)^r)} \\ &= (k-1)!^r \sum_{\lambda \vdash r(k-1)} M_{\lambda,((k-1)^r)} m_{\lambda} \end{aligned}$$

where we used **Lemma 1.1** in the first step and **Lemma 2.1** in the final step. Hence, by the interpretation of the augmented monomial symmetric functions from **Lemma 1.2**, there are,  $(1/\lambda - (1))(k - 1)!^r M_{(\lambda - (1)),((k-1)!^r)}$  ways to choose independent sets of size  $\lambda - (1)$  from  $W'_{k,r}$ . Therefore the coefficient of  $\widetilde{m}_{\lambda}$  in  $X_{W_{k,r}}$  is  $(1/\lambda - (1))(k - 1)!^r M_{(\lambda - (1)),((k-1)!^r)}$ , or, equivalently, the coefficient of  $m_{\lambda}$  in  $X_{W_{k,r}}$  is  $(\widetilde{\lambda}/\lambda - (1))(k - 1)!^r M_{(\lambda - (1)),((k-1)!^r)} = r_1(k - 1)!^r M_{(\lambda - (1)),((k-1)!^r)}$ . Conveniently, this selects all partitions such that  $r_1 > 0$ , so we don't need any restrictions in the sum over  $\lambda$ .

**Example 2.1.** We verify the formula (ii) from **Theorem 2.1** for the simple case where  $W_{k,r} = (V, E)$  and  $\lambda = (1^{\#V})$ . Since there is only one way to partition the vertex set into independent sets each of size 1, namely, by putting each vertex into its own independent set, **Lemma 1.2** tells us that the coefficient of  $\widetilde{m_{\lambda}}$  in  $X_{W_{k,r}}$ , denoted by  $[\widetilde{m_{\lambda}}]_{W_{k,r}}$ , should be exactly 1. That is,  $[m_{\lambda}]_{W_{k,r}} = (\#V)! = (r(k-1)+1)!$ .

We first compute the value of  $M_{(1)^{r(k-1)},((k-1)^r)}$ , which is the number of  $r(k-1) \times r$  0, 1matrices such that there is a single 1 in each row and k-1 1s in each column. Every such matrix is a permutation of the rows of the matrix A which has  $A_{i,j} = 1$  where  $(i-1)(k-1) < j \leq$ 



Figure 2: Lollipop graph  $L_{11,4}$ .

i(k-1) and  $A_{i,j} = 0$  otherwise. There are (r(k-1))! total such permutations. Moreover, for any  $(i-1)(k-1) < j_1 < j_2 \le i(k-1)$ , permuting rows  $j_1$  and  $j_2$  in A does not change the matrix (since the rows are identical). There are  $(k-1)!^r$  total such permutations. It follows that  $M_{(1)^{r(k-1)},((k-1)^r)} = r(k-1)!/(k-1)!^r$ .

Lastly, we note that  $r_1 = r(k-1) + 1$  where  $\lambda = ((\#V)^{r_{\#V}}, ..., 1^{r_1})$ . Finally, (ii) in **Theorem 2.1** gives the following expression.

$$[m_{\lambda}]_{W_{k,r}} = (k-1)!^r \cdot (r(k-1)+1) \frac{(r(k-1))!}{(k-1)!^r} = (r(k-1)+1)!$$

A lollipop graph on n vertices with girth c < n is the cycle graph  $C_c$  with a path of length n - c attached to any vertex in the cycle. See Figure 3 for a representation of  $L_{11,4}$ . The next theorem gives an explicit formula for the chromatic symmetric function of any, arbitrary lollipop graph. In the statement of the next theorem, we abuse the notation of the so-called "Kronecker delta" by defining for a condition S and an object O,

$$\delta_S(O) = \begin{cases} 1 & \text{if } O \text{ satisfies } S \\ 0 & \text{if } O \text{ does not satisfy } S \end{cases}$$

**Theorem 2.2.** Let  $L_{n,c}$  be the unique lollipop graph on n vertices with girth c, then,

$$X_{L_{n,c}} = \sum_{\lambda \vdash n} C_{\lambda}^{n,c} p_{\lambda}$$

where  $\lambda = (n^{r_n}, ..., 1^{r_1}),$ 

$$C_{\lambda}^{n,c} = C_{\lambda-(1)}^{n-1,c} \cdot \delta_{\{r_1>0\}}(\lambda) + \sum_{\{1 < k \le n-c: r_k > 0\}} C_{\lambda-(k)}^{n-k,c} + \sum_{\{n-c < k \le n: r_k > 0\}} (k+c-n) \cdot (-1)^{(n-k)-(\sum_{i=1}^n r_i-1)} \frac{(\sum_{i=1}^n r_i-1)! \cdot r_k}{\prod_{i=1}^n (r_i)!} + (-1)^n \cdot \delta_{\{r_n=1\}}(\lambda)$$

and,

$$C_{\lambda}^{c+1,c} = (-1)^{c - \sum_{i=1}^{c+1} r_i - 1} \frac{\left(\sum_{i=1}^{c+1} r_i - 1\right)! \cdot r_1}{\prod_{i=1}^{c+1} (r_i)!} \left(1 + \sum_{j=2}^{c} (j-1) \frac{r_j}{\sum_{i=1}^{c+1} r_i - 1}\right) \cdot \delta_{\{r_1 > 0\}}(\lambda) + (-1)^c \delta_{\{r_c = 1\}}(\lambda) + \sum_{\{1 < k \le c+1: r_k > 0\}} (k-1) \cdot (-1)^{(n-k) - (\sum_{i=1}^{c+1} r_i - 1)} \frac{\left(\sum_{i=1}^{c+1} r_i - 1\right)! \cdot r_k}{\prod_{i=1}^{c+1} (r_i)!} + (-1)^{c+1} \cdot \delta_{\{r_{c+1} = 1\}}(\lambda)$$

*Proof.* We proceed by induction on the length of the path adjoined to the cycle in  $L_{n,c}$ , that is, on n. For the base case (n = c + 1), we attach a leaf arbitrarily to a vertex in the cycle graph with c vertices to form  $L_{c+1,c}$ . Let  $e \in E(L_{c+1,c})^2$  be the unique leaf edge in  $L_{c+1,c}$ . For a fixed  $\lambda_0 \vdash n$ , we wish to apply **Lemma 1.3** by counting the number of subsets  $S \subset E(L_{c+1,c})$  such that  $\lambda_{L_{n,c}}(S) = \lambda_0$ . We divide our analysis into three disjoint cases.

In the first case, we have  $e \notin S$ , in which case the unique leaf of  $L_{c+1,c}$  will always be disconnected from the rest of the graph. Hence, if  $r_1 = 0$  in  $\lambda_0$ , there is no way to choose a subset S such that  $\lambda_{L_{n,c}}(S) = \lambda_0$ . Assume, then, that  $r_1 > 0$ , then the number of ways to choose S such that  $\lambda_{L_{n,c}}(S) = \lambda_0$  is exactly the number of ways to choose a subset  $S_0 \subset E(C_c)$  where  $C_c$  is the cycle graph on c vertices, such that  $\lambda_{C_c}(S_0) = \lambda_0 - (1)$ . Part (iv) of **Lemma 1.5** completes the argument, giving us the term,

$$(-1)^{c-\sum_{i=1}^{c+1}r_i-1} \frac{(\sum_{i=1}^{c+1}r_i-1)! \cdot r_1}{\prod_{i=1}^{c+1}(r_i)!} \left(1 + \sum_{j=2}^{c}(j-1)\frac{r_j}{\sum_{i=1}^{c+1}r_i-1}\right) \cdot \delta_{\{r_1>0\}}(\lambda) + (-1)^c \delta_{\{r_c=1\}}(\lambda)$$

In the second case,  $e \in S$  and  $S \neq E(L_{c+1,c})$ . In this case, we suppose that e is in a connected component of  $L_{c+1,c}(S)$  with vertex size k, where k > 1 is fixed. The key observation of this proof is that **the remaining graph without the connected component containing** e is a **path** (see Figure 3), so the number of ways to choose S given e is in a connected component of size k is the number of ways to choose a subset  $S_1 \subset E(P_{n-k})$  where  $P_{n-k}$  is the path graph on n - k vertices such that  $\lambda_{P_{n-k}}(S_1) = \lambda - (k)$ . Moreover, there are k - 1 ways to position the connected component containing e on the graph of  $L_{c+1,c}$ . Part (iii) of **Lemma 1.5** completes the argument, giving us the term,

$$\sum_{\{1 < k \le c+1: r_k > 0\}} (k-1) \cdot (-1)^{(n-k) - (\sum_{i=1}^{c+1} r_i - 1)} \frac{(\sum_{i=1}^{c+1} r_i - 1)! \cdot r_k}{\prod_{i=1}^{c+1} (r_i)!}$$

In the last case, S = E, from which we get the term  $(-1)^{c+1} \delta_{\{r_{c+1}=1\}}(\lambda)$ .

For the inductive step, suppose that for any  $\lambda_0 \vdash b$ ,  $C_{\lambda_0}^{b,c}$  correctly counts the number of ways to choose a subset  $S \subset E(L_{b,c})$  such that  $\lambda_{L_{n,c}}(S) = \lambda_0$ , and  $1 \leq b < n$ . Let e be the unique leaf edge in  $L_{n,c}$ . In the first case, we suppose that  $e \notin S$ , in which case there are exactly  $C_{\lambda-(1)}^{n-1,c}$  ways to choose a subset  $S \subset E(L_{n,c})$  with  $\lambda(S) = \lambda$  if  $r_1 > 0$ , and 0 otherwise.

In the second case,  $e \in S$ , and we consider the size of the connected component of  $L_{n,c}(S)$ which contains e. Suppose the vertex size of this connected component is  $1 < k \leq n - c$ , then the remaining graph besides the component containing e is exactly  $L_{n-k,c}$ , from which we conclude there are  $C_{\lambda-(k)}^{n-k,c}$  ways to choose a subset S with  $\lambda_{L_{n,c}}(S) = \lambda$ . If the size of the connected component containing e is greater than n - c, then the analysis is the same as in the base case, namely, the remaining graph of  $L_{n,c}(S)$  except for the connected component containing e is a path graph. In

<sup>&</sup>lt;sup>2</sup>For a graph G = (V, E), we let E(G) = E, that is, E(G) is the edge set of G. Similarly, we say V(G) = V is the vertex set of G.



Figure 3: Fixing the size of the connected component containing e in  $L_{9,8}$  to be 5, the remaining graph is always a path on 4 vertices, that is  $P_4$ . Moreover, there are 4 ways to position the blue edges.

this case, there are notably k - (n - c) ways to place the connected component containing e onto the graph of  $L_{n,c}$ . The result follows from Lemma 1.5.

We end with a short discussion about chromatic bases. Recently, it has been shown in [3] that there is a simple algorithm for computing the CSF of a tree in the star basis. Moreover, by determining the smallest partition of n in lexicographic order that has a nonzero coefficient in the star-basis expansion of the CSF, they show that all trees of diameter less than 5 are distinguished by the CSF. Furthermore, they prove that the set of chromatic symmetric functions of all trees is a p(n) - n + 1 dimension subspace of the set of symmetric functions. It remains to be shown what other nontrivial properties can be proven by examining the CSF in other chromatic bases, such as the path basis.

Moreover, another question about chromatic bases is which of these bases are *Schur positive*, namely, each basis element is a positive linear combination of Schur symmetric functions. For example, the basis  $\{X_{K_n} : n \ge 1\}$  is Schur positive as we have the relationship  $e_n = s_{(1^n)}$ .

## 3 Acknowledgments

Thanks to Dr. Italo Simonelli for his helpful comments and discussion.

# References

- [1] Sookin Cho and Stephanie van Willigenburg. "Chromatic bases for symmetric functions". In: Electronic Journal of Combinatorics 23.1 (2016).
- [2] Logan Crew. "A note on distinguishing trees with the chromatic symmetric function". In: *Discrete Mathematics* 345.2 (2022), p. 112682. ISSN: 0012-365X.
- [3] Michael Gonzalez, Rosa Orellana, and Mario Tomba. "The chromatic symmetric function in the star-basis". In: *arXiv preprint* (2024). arXiv: 2404.06002.
- Jeremy L. Martin, Matthew Morin, and Jennifer D. Wagner. "On distinguishing trees by their chromatic symmetric functions". In: *Journal of Combinatorial Theory, Series A* 115 (2008), p. 103143.
- [5] Richard P. Stanley. "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph". In: Advances in Mathematics 111.1 (1995), pp. 166–194.
- [6] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.